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DISSERTATION

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Some methods for generating flexible distributions .

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By

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Dedication

Praise be to Allah, by whose grace good deeds are done, and by His grace, goals are attained, and by His success, difficulties are overcome and wishes are fulfilled.

To the source of tenderness, to the one whose sincere invitation was the secret of my success...

To my mother **Hadria**, the symbol of giving and mercy, may Allah reward you with everything good for me.

To my father, **AbdelMajid**, who taught me that willpower creates miracles and that patience is the key to success.

To my beloved husband **Rafik,** thank you for your patience, support, and inclusion.

To my siblings who have always been there for me...

Abdelhakim, **Loubna** and **Somia**... Thank you for your presence in my life.

To the flower of the heart and the pulse of the soul...

To **Miral**, my precious niece, who fills my life with love and warmth.

To a little star that lights up my hope...

To **Ousaid**, my dear nephew, you are an unending joy to me.

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To my colleagues at work and study, who shared this journey with me with all its challenges and achievements ... you are an integral part of this success

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I started my university journey at the **University of 8 May 1945 in Guelma**, but I was out of school for almost twenty years. Despite the long absence, the dream remained alive within me until it was realized, thanks to God, when I had the opportunity to return to university and complete my studies at **Mohamed Cherif Messaadia University - Souk Ahras**, within the Master 2 Applied Mathematics track.

In this context, I extend my deepest thanks and gratitude to my supervisor **Dr. Benrabia Mohamed**, who was more than just an academic mentor. He provided me with invaluable support, both scientifically and humanly, and his patience, encouragement, and confidence in my ability to succeed were the most important factors that gave me the motivation and strength to continue.

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Abstract

In recent years, the use of distribution mixtures and weightings has become increasingly prevalent in research. In this dissertation, these approaches are used to introduce two novel distributions : the Mixed Gamma Lindley (MGL) and Weighted Mixed Gamma Lindley (WMGL) distributions, which are derived from Lindley and Gamma distributions with varying mixture proportions. Descriptive statistics are calculated, along with reliability analysis functions. Various estimation methods are employed to determine the distribution parameters. Additionally, the distributions of order statistics and the quantile function are computed. The Renyi and Tsallis entropy, are also assessed. Real data applications are performed for the MGL distribution, demonstrating the superior performance of this suggested distribution over their competitors, based on multiple comparison criteria.

Key words : Mixing distribution, MGL distribution, moments, reliability analysis, entropy, maximum likelihood estimation, Weighthed distribution.

Résumé

Au cours des dernières années, l'utilisation des mélanges de distributions et des pondérations est devenue de plus en plus courante dans la recherche. Dans ce mémoire, ces approches sont utilisées pour introduire deux nouvelles distributions : les distributions *Gamma-Lindley Mixte* (MGL) et *Gamma-Lindley Mixte Pondérée* (WMGL), dérivées des distributions de Lindley et Gamma avec des proportions de mélange variables. Des statistiques descriptives sont calculées, ainsi que des fonctions d'analyse de fiabilité. Diverses méthodes d'estimation sont employées pour déterminer les paramètres des distributions. De plus, les distributions des statistiques d'ordre et la fonction quantile sont évaluées. Les entropies de Rényi et de Tsallis sont également étudiées. Des applications sur des données réelles sont effectuées pour la distribution MGL, démontrant les performances supérieures de cette distribution proposée par rapport à ses concurrentes, selon plusieurs critères de comparaison.

Mots-clés : distribution de mélange, distribution MGL, moments, analyse de fiabilité, entropie, estimation du maximum de vraisemblance, distribution pondérée.

الملخص

في السنوات الأخيرة، أصبح استخدام المزائج التوزيعية والأوزان أكثر شيوعًا في الأبحاث. في هذه الأطروحة، تم استخدام هذه الأساليب لتقديم توزيعين جديدين :**توزيع غاما-ليندلي المختلط (MGL)** و**توزيع غاما-ليندلي المختلط الموزون(WMGL)** ، وهما مشتقان من توزيعي ليندلي وغاما مع نسب مزج مختلفة. تم حساب الإحصاءات الوصفية، إلى جانب تحليل الموثوقية. كما تم استخدام عدة طرق تقديرية لتحديد معلمات التوزيع. بالإضافة إلى ذلك، تم حساب توزيعات الإحصاءات المرتبة ودالة الكوانتايل. كما تم تقييم كل من إنتروبي ريني (Renyi) وتساليس .(Tsallis) وقد تم تطبيق التوزيع MGL على بيانات حقيقية، مما أظهر أداءً متفوقًا لهذا التوزيع المقترح مقارنةً بنظائره، وذلك استنادًا إلى معايير مقارنة متعددة.

الكلمات المفتاحية :توزيع المزج، توزيعMGL ، العزوم، تحليل الموثوقية، الإنتروبي، تقدير الاحتمالية العظمى، التوزيع الموزون.

Contents

Chapter 1

Introduction and Definitions

1.1 Mixture and weighted of distributions

In the field of statistics, a mixture distribution is a combination of two or more probability distributions (parent distributions). They can be either univariate or multivariate. Mathematically, a random variable X is said to have a mixture of two or more models, if its probability density function take the form

$$f\left(x\right)=\sum_{i=1}^{n}w_{i}f_{i}\left(x\right),$$

subject to $\sum_{i=1}^{n}w_{i}=1$ are the mixing weights[1], and $f_{i}\left(x\right)$ are the parent distributions. The term mixed distribution has been used by many researchers. For example, Rama distribution is a mixture of exponential (θ) and Gamma $(4,\theta)$ with mixing proportion $\frac{\theta^{3}}{\theta^{3}+6}$ and $\frac{6}{\theta^{3}+6}$ introduced by Shanker [2](2017), Shraa and Alomari[3] (2019) has proposed Darna distribution as a mixture of exponential $\left(\frac{\theta}{\alpha}\right)$ and Gamma $\left(3,\frac{\theta}{\alpha}\right)$ with proportion $\frac{2\alpha^{2}}{2\alpha^{2}+\theta^{2}}$, Garaibah [4] (2021) has suggested Garaibah distribution by mixing exp(β), Gamma $(2,\beta)$, Gamma $(4,\beta)$ and Gamma $(6,\beta)$ and mixing weights $\frac{\beta^{6}}{\beta^{6}+\beta^{4}+\beta^{2}+1}$, $\frac{\beta^{4}}{\beta^{6}+\beta^{4}+\beta^{2}+1}$, $\frac{\beta^{2}}{\beta^{6}+\beta^{4}+\beta^{2}+1}$ and $\frac{1}{\beta^{6}+\beta^{4}+\beta^{2}+1}$. Zeghdoudi and Messaadia [5] (2018) introduced the Zeghdoudi distribution as a mixture of Gamma $(2,\theta)$ and Gamma $(3,\theta)$. Another methods for generating new distribution is by certain weight, the weigthed distribution is given by:

$$f_{w}\left(x\right)=\frac{w\left(x\right)}{E\left(w\left(x\right)\right)}f\left(x\right), \tag{1.1}$$

where $w(x)$ is a positive weight and $f(x)$ is the base PDF of X. For example, Kilany [6](2016) adopted weighted Lomax distribution in function of the lomax distribution.

1.2 Moments and Associated Measures:

Statistical Moments

Consider a continuous variable X and $r \geq 1$ be an integer, the r^{th} moment of X is defined as [7]:

$$E(X^r) = \int_x x^r f(x)\, dx \tag{1.2}$$

where $f(x)$ represents the probability density function of X.

Moment generating function

The moment generating function (MGF) of a distribution in the continuous case, is expressed as follows [8]:

$$M_x(t) = E\left(e^{tx}\right) = \int_x e^{tx} f(x)\, dx \tag{1.3}$$

The moments of any distribution can be determined using two ways: by the integral formula in equation (1.2) or by differentiated the moment generating funtion and replacing t by 0.

Central moments

The central moment of random varaible X , which measures deviation around the mean $\mu = E(X)$ [9],is defined as:

$$\mu_r = E\left[(X-\mu)^r\right] = \int_x (X-\mu)^r f(x)\, dx \tag{1.4}$$

The first central moment μ_1 is zero and the second central moment corresponds to the variance of X.

Skewness and kurtosis coefficients

The skewness coefficient quantifies the symmetry of a distribution. If a distribution is symmetric, then the skewness value is 0 (Normal distribution) [10]. A positive skew indicates a longer tail on the right side while a negative skew indicates a longer tail on the left. For a random variable X, the skewness is mathematically defined as

$$SK(X) = E\left(\frac{X-\mu}{\sigma}\right)^3 \tag{1.5}$$

where μ is the mean and σ is the standard deviation [11]. This formula simplifies to:

$$SK(X) = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \tag{1.6}$$

The kurtosis coefficient measures the peakdness of a distribuition relative to the normal distribution [12]. A kurtosis value of 3 indicates tails similar to the normal distribution (Mesokurtic: Medium peaked). A kurtosis greater than 3 (leptokurtic) implies heavier tails and a sharper peak, whereas a kurtosis less than 3 (platykurtic) signifies lighter tails and a flatter peak. kurtosis is defined as:

$$Kur(X) = E\left(\frac{X-\mu}{\sigma}\right)^4, \tag{1.7}$$

simplifing this expression yields:

$$Kur(X) = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} \tag{1.8}$$

Coefficient of variation

The coefficient of variation (CV) is a statistical coefficient used to measure the relative dispersion of data points in a dataset around the mean. It represents the ratio of the standard deviation to the expected value. For a population, the CV of a random variable X, [13] is defined as:

$$CV = \frac{\sigma}{\mu}, \tag{1.9}$$

where σ is the standard deviation and μ is the population mean.

To express the CV as a percentage, the formula becomes:

$$CV = \frac{\sigma}{\mu} \times 100\%, \tag{1.10}$$

when working with sample data, the coefficient of variation is calculated using the sample standard deviation is s and the sample mean $\overline{X}$.

$$CV = \frac{s}{\overline{X}} \times 100\% \tag{1.11}$$

1.3 Reliability analysis

1.3.1 The survival function

The survival function, denoted as $S(t)$, quantifies the probability that a system component, or event will continue to operate without failure up time t [14]. It is defined as:

$$S(t) = P(X > t) = 1 - P(X \leq t) = 1 - F(t) \tag{1.12}$$

Where $F(t)$ represents the cumulative distribution function (CDF) of the failure time.

Key properties of the survival function

1. Boundary Condition
   - (a) $S(\infty) = 0$: As t approaches infinity, the probability of survival diminishes to zero.
   - (b) $S(0) = 1$: At time $t = 0$, the system is guaranteed to be functional.
2. Complementary relationship

   $F(t) + S(t) = 1$
3. Functional behavior

   $S(t)$ is a right countinuous and non increasing function, the intuitive notion that survival probability decreases or remains constant over time.

1.3.2 The hazard rate function and related functions

The hazard rate function (or failure rate function) is the instantaneous risk of failure of a component within a small time interval Δt. Consider a countinous random variable with probability density function $f(t)$ and cumulative distribution function

$F(t)$. The hazard rate function is obtained by dividing the probability density function by the reliability function expressed as[15];

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)} \tag{1.13}$$

Integration the hazard rate function over time yields the cumulative hazard rate function. we can express the formular of $H(t)$ in function of $S(t)$ by the following these steps:

$$\int_0^t h(x)\,dx = \int_0^t \frac{f(x)}{S(x)}dx \tag{1.14}$$

Substituting $f(t) = -\acute{S}(t)$, the integral simplifies to:

$$\int_0^t \frac{-\acute{S}(x)}{S(x)}dx = -\int_{S(0)}^{S(t)} \frac{dS}{S} = -\ln S(t)$$

Thus, the cumulative hazard rate function is defined as

$$H(t) = -\ln S(t) \tag{1.15}$$

The reversed hazard rate function $rh(t)$, represents the ratio of the probability density function to cumulative distribution function. Initially termed the "dual failure function", this concept has been explored in diverse fields. For instance[16] applied it in medical studies,[17] utilized it for modeling information processing capacity. Mathematically, it is defined as:

$$rh(t) = \frac{f(t)}{F(t)} \tag{1.16}$$

The odds function is used as a comparative tool between treatment and control groups in survival analysis. For a non negative random variable T with cumulative distribution function $F(t)$ and survival function $S(t)$, the odds function is formulated as[14]:

$$O(t) = \frac{F(t)}{S(t)} \tag{1.17}$$

1.4 Order statistics and quantile function

Order statistics play a crucial role in various pratical applications and statistical inference. For instance, the first and last order statistics are essential for outlier detection and reliability analysis of systems [18]. Consider a random sample (X_1,X_2,...,X_n) with probability density function $f(x)$ and cumulative distribution function $F(x)$. Order statistics are derived by arranging the sample in ascending order:

$$X_{(1)} \leq X_{(2)} \leq ... \leq X_{(k)} \leq .... \leq X_{(n)},$$

where

$$\begin{aligned} X_{(1)} &= \min(X_1, X_2, ..., X_n) \\ &\vdots \\ X_{(i)} &= i\ th \text{ smallest of } X_{1,\cdots}, X_n \\ &\vdots \\ X_{(n)} &= \max(X_1, X_2, ..., X_n) \end{aligned}$$

According to [19], the probability density function of the i^{th} order statistic $X_{(i)}$ is given by:

$$f(x_{(i)}) = i \binom{n}{i} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x). \tag{1.18}$$

The quantile function, is known as the inverse of the cumulative distribution function, it is expressed as:

$$x = F^{-1}(y) \tag{1.19}$$

1.5 Entropy

Entropy measures the average of uncertainty or randomness inherent in data [20]. The concept of entropy was formally introduced by [21]. For a continuous random variable X characterized by a probability density function $f(x)$, the entropy $S_h(X)$ is given by:

$$S_h(X) = -\int_x f(x) \ln f(x)\, dx \tag{1.20}$$

A generalized extension of Shannon entropy, known as Renyi entropy, is expressed as[22]

$$R_\lambda(X) = \frac{1}{1-\lambda} \ln \int_R [f_X(x)]^\lambda dx, \lambda > 0; \lambda \neq 0 \tag{1.21}$$

where, λ is a tunable parameter that shapes the entropy measure.

Another generalization, termed Tsallis entropy [23], is defined by:

$$T_{\lambda}(X)=\frac{1}{1-\lambda}\left(1-\int_{R}\left[f_{X}(x)\right]^{\lambda}dx\right),\lambda>0,\lambda\neq 0 \tag{1.22}$$

1.6 Methods of estimation

1.6.1 Maximum likelihood method

The maximum likelihood estimation (MLE) method is the most widely used approach in statistical inference. It is consedered as a foundational tool for various criteria and model, including the Bayesian Information Criterion (BIC), Akaike Information criterion (AIC), chi-square tests, and Bayesian methods.

Consider a random sample $X_1, X_2, \ldots, X_n$ of size n drawn from a probability density function $f(x_i \mid \Theta)$, where Θ represents the parameter space. The likelihood function is defined as[24]:

$$L(x_i \mid \Theta)=\prod_{i=1}^{n} f(x_i,\Theta). \tag{1.23}$$

The maximum likelihood estimator is a derivation of the ln −likelihood function with respect to the parameters.

1.6.2 Ordinary and weighted least squares method of estimations

The least squares methods are commonly employed for parameter estimation in linear regression models. Let $X_1, X_{2,}, \ldots, X_n$ be a radom sample with cumulative distribution function , and let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the ordered observations.

Ordinary least squares (OLS)

The OLS estimates are obtained by minimizing the following sum of squared deviations [25]:

$$\sum_{i=1}^{n}\left[F(x_{(i)})-\frac{i}{n+1}\right]^{2}, \tag{1.24}$$

where

$$E\left[F(x_i)\right]=\frac{i}{n+1}$$

Weighted least squares(WLS)

The WLS estimates are derived by minimizing the weighted sum of squared deviations in the following equation:

$$\sum_{i=1}^{n} w_j \left[F\left(x_{(i)}\right) - \frac{i}{n+1} \right]^2, \tag{1.25}$$

where the weight w_j is inversely proportional to the variance of $F\left(x_{(i)}\right)$:

$$w_j = \frac{1}{Var\left[F\left(x_{(i)}\right)\right]} = \frac{(n+1)^2 (n+2)}{i\,(n-i+1)}.$$

Here, the variance term is given by:

$$Var\left[F\left(x_{(i)}\right)\right] = \frac{i\,(n-i+1)}{(n+1)^2 (n+2)}$$

1.7 Statistical tests

Akaike information criterion test

The Akaike information criterion (AIC) is a statistical measure used to evaluate and compare the relative quality of candidate models for a given dataset [26]. Let k denote the number of parameters estimated within a model, and let $\hat{L}$ represent the maximum value of the model's likelihood function. The AIC value is computed as:

$$AIC = 2k - 2\ln\left(\hat{L}\right) \tag{1.26}$$

Corrected Akaike information criterion test

The corrected Akaike information criterion $(CAIC)$ is an adjusted version of the Akaike information criterion (AIC), desined to address overfitting in statistical models, particularly when the sample size is small or the number of estimated parameters is large. The $CAIC$ introduces an additional penalty term to the AIC formula to account for these conditions, ensuring a more conservative model selection. The mathematical expression for $CAIC$ is defined as [27]:

$$CAIC = AIC + \frac{2k\,(k+1)}{n-k-1} \tag{1.27}$$

Bayesian information criterion test

The Bayesian information criterion (BIC) is a statistical tool employed to compare and select models based on their fit to observed data, while penalizing model complexity to prevent overfitting. Mathematically, the BIC is expressed as[28]:

$$BIC = k \ln(n) - 2 \ln\left(\hat{L}\right),$$

where n is the sample size, k and $\hat{L}$ as previously stated in (1.26)

Hannan-Quinn information criterion test

The Hannan-Quinn information criterion ($HQIC$) is a statistical measure used to evaluate the appropriateness of a model. It balances model fit and complexity by penalizing the number of parameters. The $HQIC$ is defined as[29]:

$$HQIC = -2 \ln\left(L\left(\hat{\theta}\right)\right) + 2 \ln(\ln(n)) \tag{1.28}$$

Kolmogorov-Smirnov test

The Kolmogorov-smirnov (KS) test, initially introduced by kolmogorov and later refined as statistical test by smirnev[31], is a non-parametric method designed for analyzing continuous, one-dimensional data. This test evaluates the maximum absolute discrepancy between two cumulative distribution functions (CDF_S)[32]. The KS statistic is mathematically expressed as:

$$KS = \sup\left|F_n(x) - F(x)\right|,$$

where $F(x)$ represents the theoretical cumulative distribution function, and $F_n(x)$, the empirical ccumulative distibution function, is defined as:

$$F_n(x) = \frac{\text{number of (elements in the sample } \leq x)}{n}$$

The optimal model is indentified by minimizing the KS value, as the smallest discrepancy indicates the closest alignment between the empirical data and the theoretical distribution.

1.8 Dissertation structure

This dissertation is structured into three chapters. In the first chapter, we presented the theoretical foundation, starting with an introduction to mixture distributions and then we touched on basic statistical definitions and concepts. The second chapter is dedicated to the Mixed Gamma Lindley distribution, along with some of its propreties and statistical applications. In the third chapter, we study the weighted Mixed Gamma Lindley distribution. Finally, we conclude with a conclusion.

Chapter 2

Mixed Gamma Lindley distribution with application to real data

2.1 Introduction

The Lindley distribution is a continuous probability model that exhibits haracteristics similar to the exponential distribution, and it has been employed in various applied statistical contexts. It is defined for a non-negative random variable and is parameterized by a positive parameter θ. The distribution is characterized by the following key function:

The cumulative distribution function (CDF) is given by:

$$K\left(x\right)=1-\frac{\left(1+\theta+\theta x\right)e^{-\theta x}}{1+\theta},x>0,\theta>0 \tag{2.1}$$

The probability density function (CDF) is expressed as:

$$k\left(x\right)=\frac{\theta^{2}\left(1+x\right)e^{-\theta x}}{1+\theta},x>0,\theta>0 \tag{2.2}$$

The moment generating function (MGF) takes the form:

$$M_{X}\left(t\right)=\frac{-\theta^{2}}{\left(\theta+1\right)\left(\theta-t\right)} \tag{2.3}$$

On the other hand, the Gamma distribution represents a fiexible two parameter

family of continuous distributions frequently utilized in statistical modeling. It generalizes several notable distribution, including the exponential, and chi-squared distribution. The Gamma distribution can be parameterized either in terms of a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$, or equivalently, in terms of a shape parameter and a rate parameter $\lambda = 1/\beta$. The distribution is described by:
The CDF:

$$Y(x) = \frac{\gamma(\beta, \theta x)}{\Gamma(\beta)} \tag{2.4}$$

The PDF:

$$y(x) = \frac{\theta^{\beta} x^{\beta-1} e^{-\theta x}}{\Gamma(\beta)}, x > 0, \theta, \beta > 0 \tag{2.5}$$

The MGF:

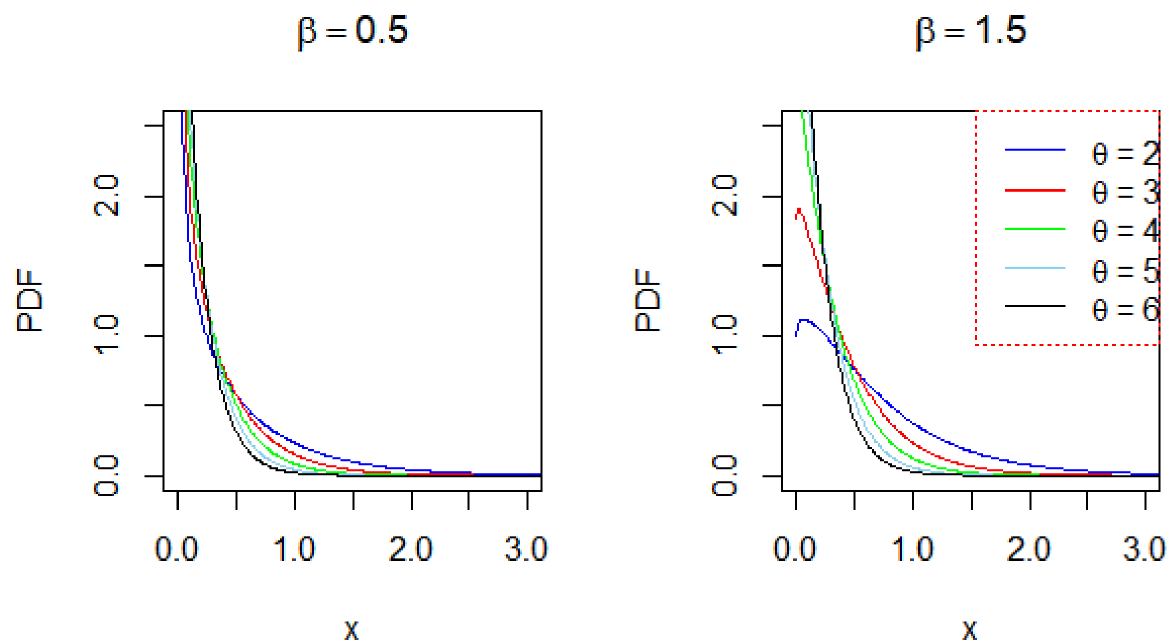
$$M_X(t) = \frac{\theta^{\beta}}{(\theta - t)^{\beta}} \tag{2.6}$$

2.2 Mixed Gamma Lindley Distribution

The MGL distribution is formulated by blending two distinct distributions: The Lindley distribution with parameter θ and the Gamma distribution with parametrs θ and β. The mixing proportions used in mixture are: $p_1 = \frac{\theta\beta}{\theta\beta+1}$ and $p_2 = \frac{1}{\theta\beta+1}$
The combined PDF of the MGL distribution integrates these two components as follows:

$$f(x; \beta, \theta) = \frac{\theta^3 \beta (1 + x) e^{-\theta x}}{(\theta\beta + 1)(\theta + 1)} + \frac{\theta^{\beta} x^{\beta-1} e^{-\theta x}}{(\theta\beta + 1)\Gamma(\beta)}$$

$$f(x; \beta, \theta) = \frac{\theta^3 \beta \Gamma(\beta)(1 + x) + \theta^{\beta} x^{\beta-1}(\theta + 1)}{\Gamma(\beta)(\theta\beta + 1)(\theta + 1)} e^{-\theta x} \tag{2.7}$$



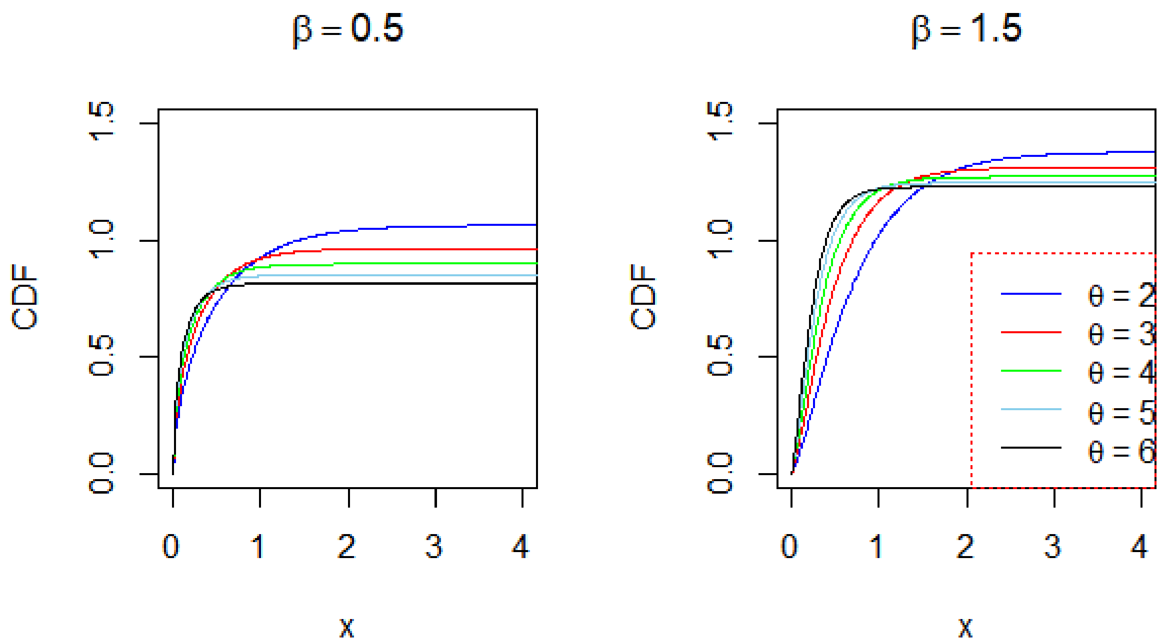
The pdf of MGL distribution for different values θ and β.

We observe that the MGL model is a decreasing function for a variate values of θ and β and it is right skewed. Also, we can see that, as the value of θ increase, the distribution exhibits heavier tail.

The cumulative distribution function (CDF) of the MGL distribution is derived by integrating the PDF over the interval $[0, x]$:

$$\begin{aligned} F(x;\beta,\theta) &= \int_0^x \left(\frac{\theta^3 \beta (1+t) e^{-\theta t}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta t^{\beta-1} e^{-\theta t}}{(\theta\beta+1)\Gamma(\beta)} \right) dt \\ &= \int_0^x \frac{\theta^3 \beta (1+t) e^{-\theta t}}{(\theta\beta+1)(\theta+1)} dt + \int_0^x \frac{\theta^\beta t^{\beta-1} e^{-\theta t}}{(\theta\beta+1)\Gamma(\beta)} dt \\ &= \frac{\theta\beta}{\theta\beta+1} \left(1 - \frac{(1+\theta+\theta x) e^{-\theta x}}{1+\theta} \right) + \frac{1}{\theta\beta+1} \frac{\gamma(\beta,\theta x)}{\Gamma(\beta)} \\ &= \frac{1}{\theta\beta+1} \left(\theta\beta - \frac{(\theta\beta+\theta^2\beta+\theta^2\beta x) e^{-\theta x}}{1+\theta} + \frac{\gamma(\beta,\theta x)}{\Gamma(\beta)} \right), x>0, \theta, \beta>0 \end{aligned} \tag{2.8}$$

Where $\gamma(\beta, \theta x)$ is the lower incomplete Gamma function.



The cdf of MGL distribution for different values of θ and β.

2.3 Statistical Properties of the proposed model

This section derives some statistial properties of the Mixed Gamma Lindley distribution , focusing on its moments, moment generating function, skewness, kurtosis and the coefficient of variation.

Moments

Consider a random variable X following the MGL distribution with parameters β and θ. The rth moment, is derived using the Gamma function $\Gamma(r)$ =

$\int_0^\infty x^{r-1} e^{-x} dx$. The expression for $E(X^r)$ is formulated as :

$$\begin{aligned} E\left(X^r\right) &= \int_0^\infty x^r \left(\frac{\theta^3 \beta (1+x) e^{-\theta x}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta x^{\beta-1} e^{-\theta x}}{(\theta\beta+1)\Gamma(\beta)} \right) dx \\ &= \frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)} \int_0^\infty x^r (1+x) e^{-\theta x} dx + \frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)} \int_0^\infty x^r \left(x^{\beta-1} e^{-\theta x} \right) dx \\ &= \frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)} \left(\int_0^\infty x^r e^{-\theta x} dx + \int_0^\infty x^{r+1} e^{-\theta x} dx \right) \\ &+ \frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)} \left(\int_0^\infty x^{(r+\beta)-1} e^{-\theta x} dx \right) \\ &= \frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)} \left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}} \right) + \frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)} \frac{\Gamma(r+\beta)}{\theta^{r+\beta}} \\ &= \frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)} \left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}} \right) + \frac{\Gamma(r+\beta)}{(\theta\beta+1)\Gamma(\beta)\theta^r} \qquad (2.9) \end{aligned}$$

Remplacing r=1,r=2, r=3, and r=4 in (2.9), the first four moments of the MGL distribution are

$$E(X) = \frac{\theta^2\beta + 3\theta\beta + \beta}{\theta^3\beta + \theta^2\beta + \theta^2 + \theta} \qquad (2.10)$$

$$E\left(X^2\right) = \frac{2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta}{\theta^4\beta + \theta^3\beta + \theta^3 + \theta^2} \qquad (2.11)$$

$$E\left(X^3\right) = \frac{6\theta^2\beta + 26\theta\beta + \theta\beta^3 + 3\beta^2\theta + \beta^3 + 3\beta^2 + 2\beta}{\theta^5\beta + \theta^4\beta + \theta^4 + \theta^3} \qquad (2.12)$$

$$E\left(X^4\right) = \frac{\begin{array}{c} 24\theta^2\beta + 126\theta\beta + \beta^4\theta + 6\beta^3\theta + 11\theta\beta^2 + \\ \beta^4 + 6\beta^3 + 11\beta^2 + 6\beta \end{array}}{\theta^6\beta + \theta^5\beta + \theta^5 + \theta^4} \qquad (2.13)$$

Based on these moments, the variance of MGL distribution is defined using equations (1.4), (2.10), and (2.11) as

$$\begin{aligned} V(X) = \sigma_X^2 &= \left(\frac{2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta}{\theta^4\beta + \theta^3\beta + \theta^3 + \theta^2} \right) - \left(\frac{\theta^2\beta + 3\theta\beta + \beta}{\theta^3\beta + \theta^2\beta + \theta^2 + \theta} \right)^2 \\ &= \frac{(\theta^2\beta + \theta\beta + \theta + 1)(2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta) - (\theta^2\beta + 3\theta\beta + \beta)^2}{(\theta^3\beta + \theta^2\beta + \theta^2 + \theta)^2} \end{aligned}$$

Therefore, the standard deviation of the mixed Gamma Lindley distribution is

$$\sigma_X = \sqrt{\frac{(\theta^2\beta + \theta\beta + \theta + 1)(2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta) - (\theta^2\beta + 3\theta\beta + \beta)^2}{(\theta^3\beta + \theta^2\beta + \theta^2 + \theta)^2}}$$

$$\sigma_X = \frac{\sqrt{\left(\theta^2\beta + \theta\beta + \theta + 1\right)\left(2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta\right) - \left(\theta^2\beta + 3\theta\beta + \beta\right)^2}}{\theta^3\beta + \theta^2\beta + \theta^2 + \theta} \tag{2.14}$$

The coefficient of variation given by equations (1.9), (2.10), and (2.14) as

$$CV = \frac{\frac{\sqrt{\left(\theta^2\beta+\theta\beta+\theta+1\right)\left(2\theta^2\beta+7\theta\beta+\beta^2\theta+\beta^2+\beta\right)-\left(\theta^2\beta+3\theta\beta+\beta\right)^2}}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}}{\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}}$$

After some calculations, it becomes

$$CV = \frac{\sqrt{\left(\theta^2\beta + \theta\beta + \theta + 1\right)\left(2\theta^2\beta + 7\theta\beta + \beta^2\theta + \beta^2 + \beta\right) - \left(\theta^2\beta + 3\theta\beta + \beta\right)^2}}{\theta^2\beta + 3\theta\beta + \beta}$$

Moment generating function

The moment generating function (MGF) of the MGL distribution is derivad by combining equtions (1.3)and (2.7). It is expressed as:

$$\begin{aligned}
M_X(t) &= \int_0^\infty e^{tx}\left(\frac{\theta^3\beta\left(1+x\right)e^{-\theta x}}{\left(\theta\beta+1\right)\left(\theta+1\right)} + \frac{\theta^\beta x^{\beta-1}e^{-\theta x}}{\left(\theta\beta+1\right)\Gamma\left(\beta\right)}\right)dx \\
&= \int_0^\infty \frac{\theta^3\beta\left(1+x\right)e^{-x(\theta-t)}}{\left(\theta\beta+1\right)\left(\theta+1\right)}dx + \int_0^\infty \frac{\theta^\beta x^{\beta-1}e^{-x(\theta-t)}}{\left(\theta\beta+1\right)\Gamma\left(\beta\right)}dx \\
&= \frac{\theta^3\beta}{\left(\theta\beta+1\right)\left(\theta+1\right)}\int_0^\infty \left(1+x\right)e^{-x(\theta-t)}dx + \frac{\theta^\beta}{\left(\theta\beta+1\right)\Gamma\left(\beta\right)}\int_0^\infty x^{\beta-1}e^{-x(\theta-t)}dx \\
&= \frac{\theta^3\beta}{\left(\theta\beta+1\right)\left(\theta+1\right)}\left(\frac{1}{\theta-t} + \frac{1}{\left(\theta-t\right)^2}\right) + \frac{\theta^\beta}{\left(\theta\beta+1\right)\left(\theta-t\right)^\beta}, t < \theta
\end{aligned}$$

Skewness

Using (1.6), (2.10), (2.11) ,and (2.12), the skewness is given by the following formula

$$SK(X) = \frac{\left(\frac{6\theta^2\beta+26\theta\beta+\theta\beta^3+3\beta^2\theta+\beta^3+3\beta^2+2\beta}{\theta^5\beta+\theta^4\beta+\theta^4+\theta^3}\right) + 2\left(\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}\right)^3 - 3\left(\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}\right)\left(\frac{2\theta^2\beta+7\theta\beta+\beta^2\theta+\beta^2+\beta}{\theta^4\beta+\theta^3\beta+\theta^3+\theta^2}\right)}{\left(\frac{\left(\theta^2\beta+\theta\beta+\theta+1\right)\left(2\theta^2\beta+7\theta\beta+\beta^2\theta+\beta^2+\beta\right)-\left(\theta^2\beta+3\theta\beta+\beta\right)^2}{\left(\theta^3\beta+\theta^2\beta+\theta^2+\theta\right)^2}\right)^{\frac{3}{2}}}$$

.

Kurtosis

Kurtosis is derived using equations (1.8) , (2.10), (2.11), (2.12) ,and (2.13) as follows:

$$Kur = \frac{\left(\frac{\begin{array}{c}24\theta^2\beta+126\theta\beta+\beta^4\theta+6\beta^3\theta+11\theta\beta^2+\\ \beta^4+6\beta^3+11\beta^2+6\beta\end{array}}{\theta^6\beta+\theta^5\beta+\theta^5+\theta^4}\right) - 4\left(\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}\right)\left(\frac{6\theta^2\beta+26\theta\beta+\theta\beta^3+3\beta^2\theta+\beta^3+3\beta^2+2\beta}{\theta^5\beta+\theta^4\beta+\theta^4+\theta^3}\right) + 6\left(\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}\right)^2\left(\frac{2\theta^2\beta+7\theta\beta+\beta^2\theta+\beta^2+\beta}{\theta^4\beta+\theta^3\beta+\theta^3+\theta^2}\right) - 3\left(\frac{\theta^2\beta+3\theta\beta+\beta}{\theta^3\beta+\theta^2\beta+\theta^2+\theta}\right)^4}{\left(\frac{(\theta^2\beta+\theta\beta+\theta+1)(2\theta^2\beta+7\theta\beta+\beta^2\theta+\beta^2+\beta)-(\theta^2\beta+3\theta\beta+\beta)^2}{(\theta^3\beta+\theta^2\beta+\theta^2+\theta)^2}\right)^2}$$

Tabel 1: Mean, standard deviation, skewness, kurtosis and C.V of variation for MGL distribution for different values of θ and β

β	θ	μ	σ	$Skewness$	$Kurtosis$	$C.V.$
3	$0,05$	$0,0688$	$0,2093$	$4,8871$	$32,0148$	$304,0539$
3	$0,24$	$0,2209$	$0,3316$	$2,5643$	$9,1074$	$150,0865$
3	$0,43$	$0,2973$	$0,3595$	$2,1896$	$6,8315$	$120,9076$
3	$0,62$	$0,3432$	$0,3715$	$2,0379$	$6,0155$	$108,2247$
3	$0,81$	$0,3739$	$0,3793$	$1,95$	$5,5562$	$101,4317$
3	1	$0,3958$	$0,3858$	$1,8875$	$5,228$	$97,475$
4	$0,05$	$0,0604$	$0,1681$	$4,4353$	$26,4093$	$278,2013$
4	$0,24$	$0,1776$	$0,2523$	$2,4436$	$8,3514$	$142,1273$
4	$0,43$	$0,2292$	$0,2688$	$2,1479$	$6,6348$	$117,2567$
4	$0,62$	$0,2583$	$0,2757$	$2,0307$	$6,0132$	$106,726$
4	$0,82$	$0,277$	$0,2804$	$1,9614$	$5,6485$	$101,214$
4	1	$0,29$	$0,2844$	$1,9112$	$5,3792$	$98,0791$
β	θ	μ	σ	$Skewness$	$Kurtosis$	$C.V.$
5	$0,05$	$0,0547$	$0,1415$	$4,1082$	$22,7025$	$258,8585$
5	$0,24$	$0,1491$	$0,2034$	$2,3632$	$7,8779$	$136,4154$
5	$0,43$	$0,1866$	$0,214$	$2,1223$	$6,5226$	$114,7223$
5	$0,62$	$0,2067$	$0,2185$	$2,0276$	$6,0246$	$105,7215$
5	$0,81$	$0,2192$	$0,2216$	$1,9706$	$5,722$	$101,0898$
5	1	$0,2278$	$0,2244$	$1,9287$	$5,4938$	$98,5016$
6	$0,05$	$0,0504$	$0,1228$	$3,858$	$20,0637$	$243,7225$
6	$0,24$	$0,1288$	$0,1702$	$2,306$	$7,5549$	$132,0997$
6	$0,43$	$0,1573$	$0,1775$	$2,1048$	$6,45$	$112,852$
6	$0,62$	$0,172$	$0,1806$	$2,026$	$6,037$	$104,9938$
6	$0,82$	$0,181$	$0,1828$	$1,9775$	$5,7786$	$101,0027$
6	1	$0,1871$	$0,1848$	$1,9417$	$5,5807$	$98,8028$

From the table, we can see that as θ increases, the mean and the standard deviation increase, and the values of μ and σ decrease as β increases.

The positive skewness indicate that the distribution is right skewed as shown in figures above.

2.4 Reliability Analysis

This section presents the reliability-related functions of the MGL distribution, including the survival function, hazard rate function, reverse hazard function, cumulative hazard function, and odds rate function.

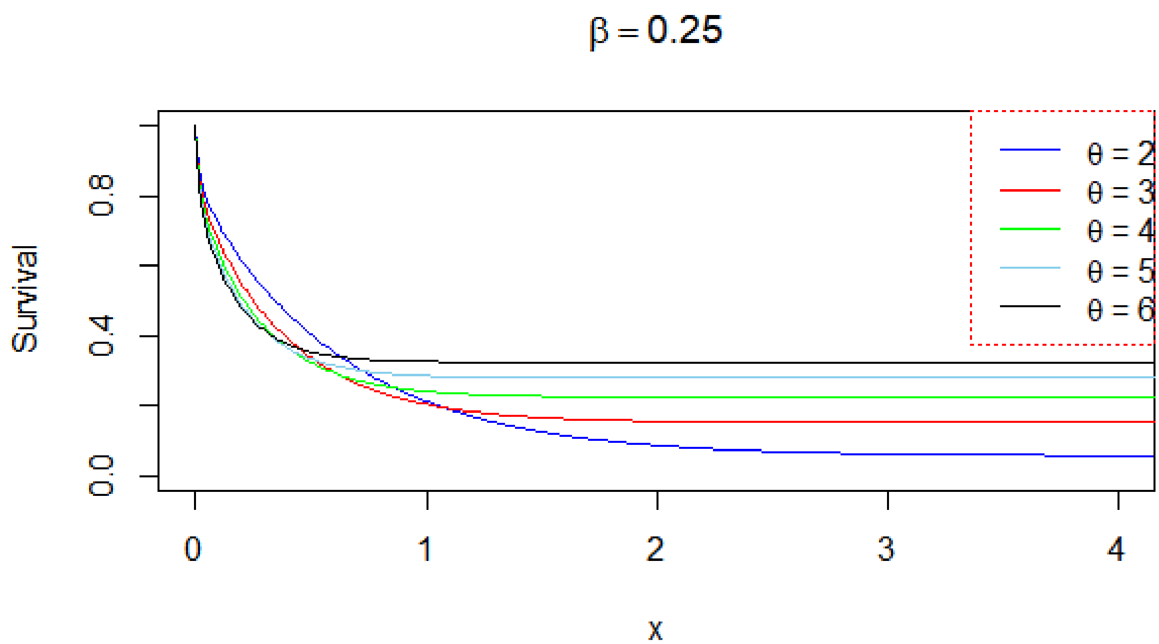
2.4.1 Survival function

The survival (or reliability) function of the MGL distribution is obtained by combining equations (1.12)and (2.8) :

$$S(t) = 1 - \left(\frac{\theta\beta}{\theta\beta+1} - \frac{(\theta\beta+\theta^2\beta+\theta^2\beta t)\,e^{-\theta t}}{(\theta\beta+1)(\theta+1)} + \frac{\gamma(\beta,\theta t)}{(\theta\beta+1)\,\Gamma(\beta)} \right),$$

simplifying the expression yields:

$$\begin{aligned} S(t) &= \frac{1}{\theta\beta+1}\left(\theta\beta+1-\theta\beta+\frac{(\theta\beta+\theta^2\beta+\theta^2\beta t)\,e^{-\theta t}}{\theta+1} - \frac{\gamma(\beta,\theta t)}{\Gamma(\beta)} \right) \\ &= \frac{(\theta+1)\,\Gamma(\beta)+\Gamma(\beta)\,(\theta\beta+\theta^2\beta+\theta^2\beta t)\,e^{-\theta t}-(\theta+1)\,\gamma(\beta,\theta t)}{(\theta\beta+1)(\theta+1)\,\Gamma(\beta)} \end{aligned} \tag{2.15}$$

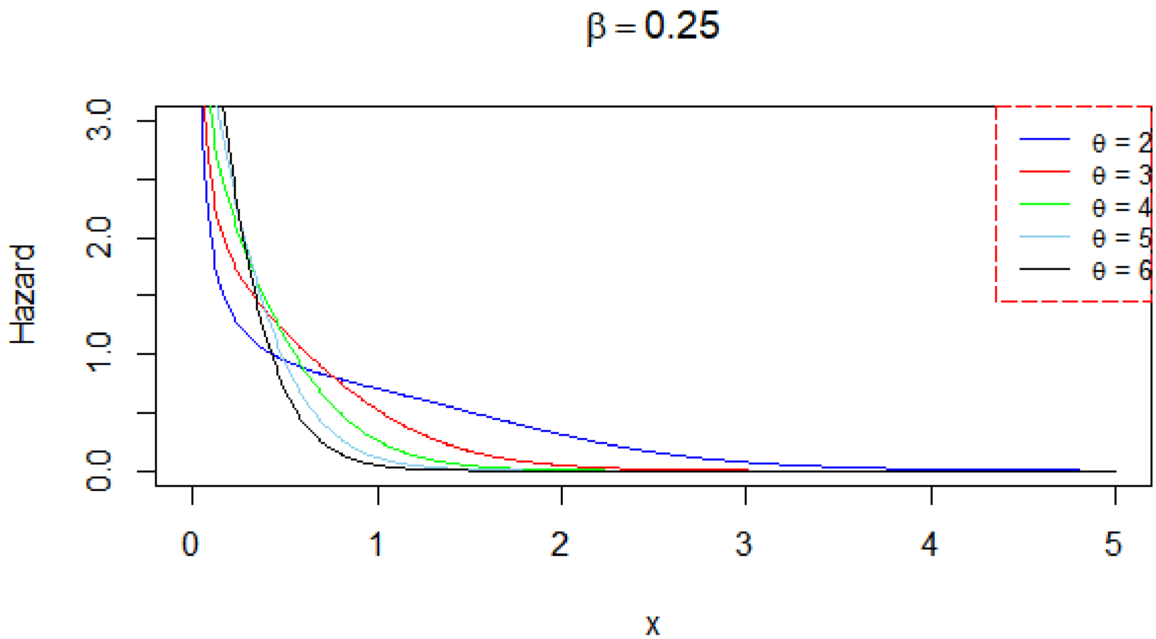


The survival function of MGL distribution for different values of θ and β.

2.4.2 Hazard rate function

The hazard rate function $h(t)$ is derived by substituting the probability density function $f(t,\beta,\theta)$ and the survival function $S(t)$ into equation (1.13) :

$$\begin{aligned} h(t) &= \frac{\frac{\theta^3\beta(1+t)e^{-\theta t}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta t^{\beta-1}e^{-\theta t}}{(\theta\beta+1)\Gamma(\beta)}}{\frac{(\theta+1)\Gamma(\beta)+\Gamma(\beta)(\theta\beta+\theta^2\beta+\theta^2\beta t)e^{-\theta t}-(\theta+1)\gamma(\beta,\theta t)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}} \\ &= \frac{\theta^3\beta\Gamma(\theta)(1+t)e^{-\theta t}+\theta^\beta(\theta+1)x^{\beta-1}e^{-\theta t}}{(\theta+1)\Gamma(\beta)+\Gamma(\beta)(\theta\beta+\theta^2\beta+\theta^2\beta t)e^{-\theta t}-(\theta+1)\gamma(\beta,\theta t)} \end{aligned}$$

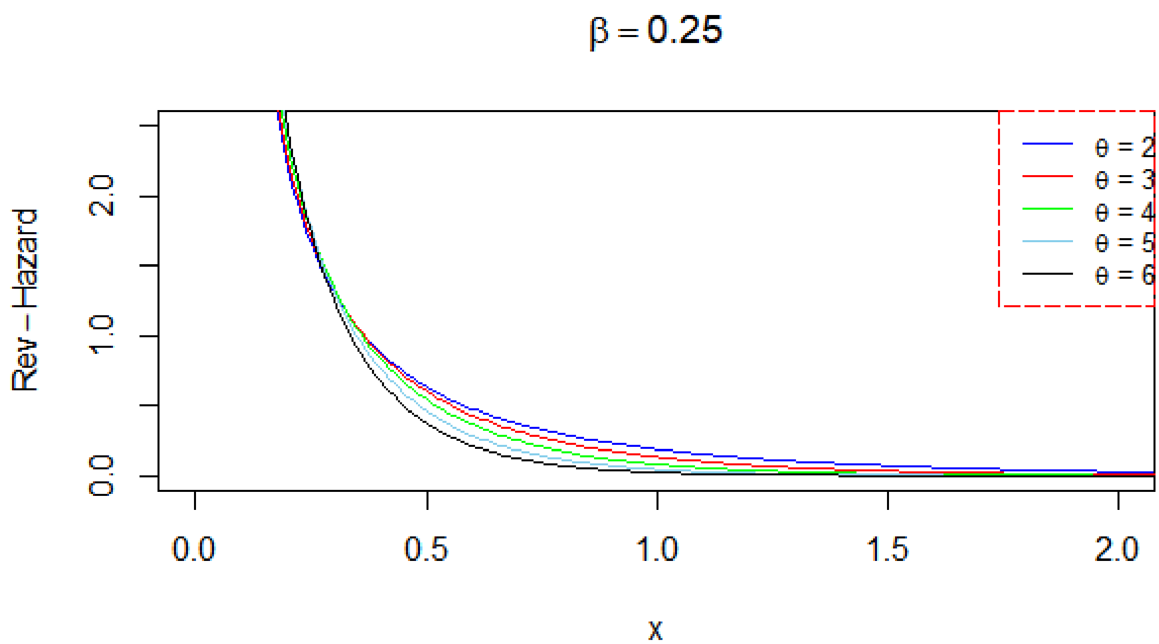


The hazard rate function of MGL distribution for different values of θ and β.

2.4.3 Reversed hazard rate function

The reversed hazard rate of MGL distribution can be derived using equations (1.16) , (2.7) and (2.8) as follows:

$$rh(t) = \frac{\frac{\theta^3\beta(1+t)e^{-\theta t}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta t^{\beta-1}e^{-\theta t}}{(\theta\beta+1)\Gamma(\beta)}}{\frac{\theta\beta}{\theta\beta+1} - \frac{(\theta\beta+\theta^2\beta+\theta^2\beta t)e^{-\theta t}}{(\theta\beta+1)(\theta+1)} + \frac{\gamma(\beta,\theta t)}{(\theta\beta+1)\Gamma(\beta)}}$$
$$= \frac{\theta^3\beta\Gamma(\beta)(1+t)e^{-\theta t} + \theta^\beta(\theta+1)t^{\beta-1}e^{-\theta t}}{\theta\beta(\theta+1)\Gamma(\beta) - \Gamma(\beta)(\theta\beta+\theta^2\beta+\theta^3\beta t)e^{-\theta t} + (\theta+1)\gamma(\beta,\theta t)}$$

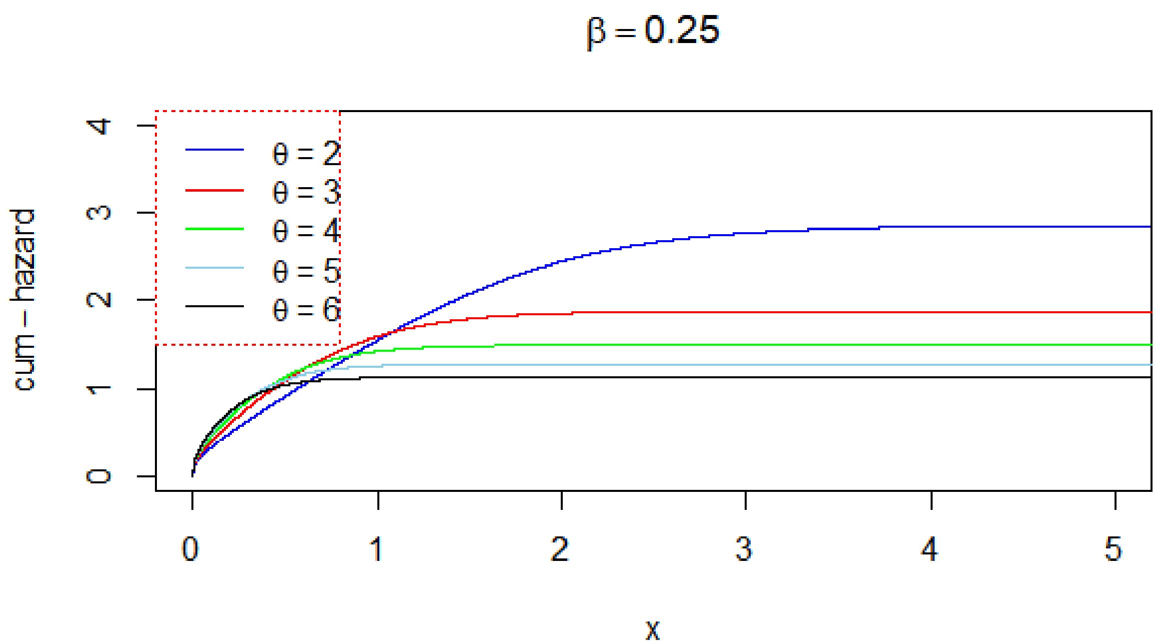


The reversed hazard rate function of MGLD for different values of θ and β.

2.4.4 Cumulative hazard rate function

Using equations (1.15) and (2.15), the cumulative hazard function $H(t)$ for the MGL distribution is derived as follows:

$$\begin{aligned} H(t) &= -\ln\left[\frac{(\theta+1)\Gamma(\beta)+\Gamma(\beta)(\theta\beta+\theta^2\beta+\theta^2\beta t)e^{-\theta t}-(\theta+1)\gamma(\beta,\theta t)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\right] \\ &= -\ln\left[(\theta+1)\Gamma(\beta)+\Gamma(\beta)\left(\theta\beta+\theta^2\beta+\theta^2\beta t\right)e^{-\theta t}-(\theta+1)\gamma(\beta,\theta t)\right] \\ &+\ln\left[(\theta\beta+1)(\theta+1)\Gamma(\beta)\right] \end{aligned}$$



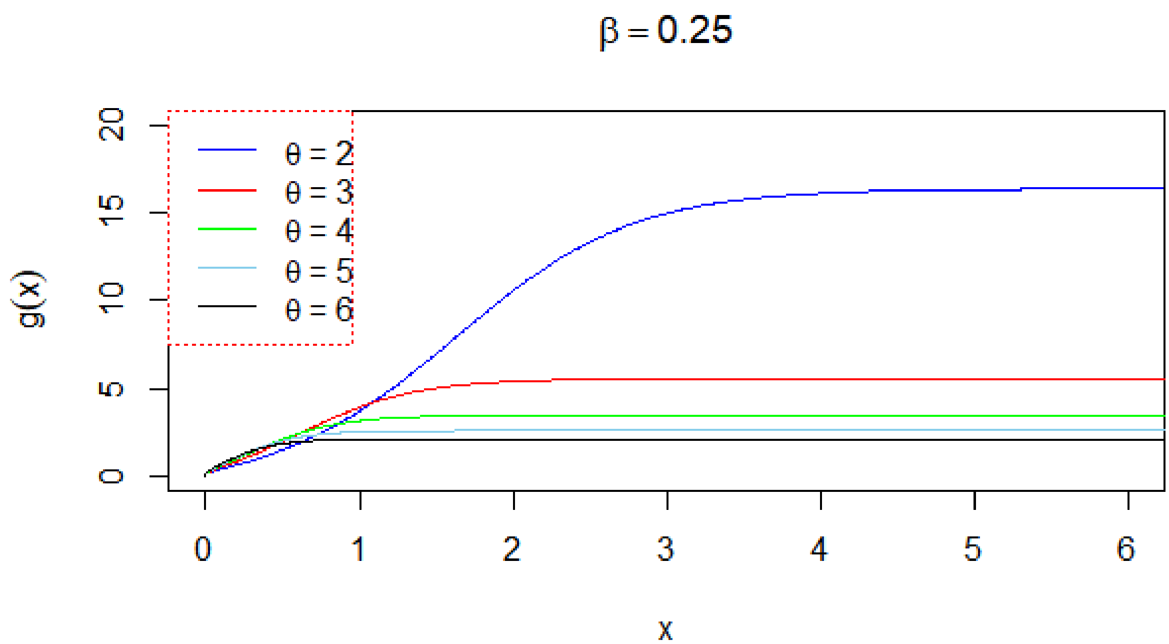
The cumulative hazard rate function of MGLD for different values of θ and β.

2.4.5 Odds rate function

The odds rate function for the Mixed Gamma Lindley distribution can be derived by substituting equations (2.8) and (2.15) into the general formula provided in (1.17). This gives the following expression:

$$O(t) = \frac{\frac{\theta\beta(1+\theta)\Gamma(\beta)-\Gamma(\beta)\left(\theta\beta+\theta^2\beta+\theta^2\beta t\right)e^{-\theta t}+(\theta+1)\gamma(\beta,\theta t)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}}{\frac{(\theta+1)\Gamma(\beta)+\Gamma(\beta)(\theta\beta+\theta^2\beta+\theta^2\beta t)e^{-\theta t}-(\theta+1)\gamma(\beta,\theta t)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}}$$

$$= \frac{\theta\beta\,(1+\theta)\,\Gamma\,(\beta) - \Gamma\,(\beta)\,(\theta\beta+\theta^2\beta+\theta^2\beta t)\,e^{-\theta t} + (\theta+1)\,\gamma\,(\beta,\theta t)}{(\theta+1)\,\Gamma\,(\beta) - \Gamma\,(\beta)\,(\theta\beta+\theta^2\beta+\theta^2\beta t)\,e^{-\theta t} - (\theta+1)\,\gamma\,(\beta,\theta t)}$$



The odds rate function of MGLD for different values of θ and β.

From pictures the figures above, we can deduce that the hazard, the survival, and the reversed hazard are a decreasing functions, but the odds and the cumulative hazard are increasing function.

2.5 Order Statistics and Quantile Function

This section focuses on deriving the distribution of order statistic and the quantile function for the Mixed Gamma Lindley distribution .

2.5.1 Order statistics

Consider a random sample $X_{1,}X_2, ..., X_n$ drawn from the MGL distribution, and let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the corresponding order statistics, where $X_{(i)}$ represents the i^{th} smallest value in the sample. The pdf of the i^{th} order statistic, $X_{(i)}$, can be derived using the standard formula for order statistics. Specifically, this involves substituting the cumulative distribution function (CDF) and probability density function (pdf) of MGL distribution into the general expression of order statistics.

For a continuous distribution , the pdf of $X_{(i)}$ is obtained using (1.18), (2.7) and (2.8)

$$f_{X_{(i)}}(x) = i\binom{n}{i}\left[\frac{\theta\beta}{\theta\beta+1} - \frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)(\theta\beta+1)} + \frac{\gamma(\beta,\theta x)}{(\theta\beta+1)\Gamma(\beta)}\right]^{i-1}$$
$$\times\left[1-\left(\frac{\theta\beta}{\theta\beta+1} - \frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)(\theta\beta+1)} + \frac{\gamma(\beta,\theta x)}{(\theta\beta+1)\Gamma(\beta)}\right)\right]^{n-i}$$
$$\times\left[\frac{\theta^3\beta(1+x)e^{-\theta x}}{(\theta\beta+1)(1+\theta)} + \frac{\theta^\beta x^{\beta-1}e^{-\theta x}}{(\theta\beta+1)\Gamma(\beta)}\right]$$
$$= i\binom{n}{i}\left(\frac{1}{\theta\beta+1}\right)^{i-1}\left[\frac{\theta\beta(\theta+1)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)} + \frac{\gamma(\beta,\theta x)}{\Gamma(\beta)}\right]^{i-1}$$
$$\times\left[1-\frac{\theta\beta}{\theta\beta+1} + \frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)(\theta\beta+1)} - \frac{\gamma(\beta,\theta x)}{(\theta\beta+1)\Gamma(\beta)}\right]^{n-i}$$
$$\times\left[\frac{\theta^3\beta(1+x)e^{-\theta x}}{(\theta\beta+1)(1+\theta)} + \frac{\theta^\beta x^{\beta-1}e^{-\theta x}}{(\theta\beta+1)\Gamma(\beta)}\right]$$
$$= i\binom{n}{i}\left(\frac{1}{\theta\beta+1}\right)^{i-1}\left[\sum_{k=0}^{i-1}\binom{i-1}{k}\left(\frac{\theta\beta(\theta+1)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)}\right)^{k} \times\left(\frac{\gamma(\beta,\theta x)}{\Gamma(\beta)}\right)^{i-1-k}\right]$$
$$\times\left(\frac{1}{\theta\beta+1}\right)^{n-i}\left[\sum_{k=0}^{n-i}\binom{n-i}{k}\left(\frac{\Gamma(\beta)-\gamma(\beta,\theta x)}{\Gamma(\beta)}\right)^{k}\times\left(\frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)}\right)^{n-i-k}\right]$$
$$\times\left(\frac{1}{\theta\beta+1}\right)\left[\frac{\theta^3\beta(1+x)e^{-\theta x}}{1+\theta} + \frac{\theta^\beta x^{\beta-1}e^{-\theta x}}{\Gamma(\beta)}\right]$$
$$= i\binom{n}{i}\left(\frac{1}{\theta\beta+1}\right)^{n}\left[\sum_{k=0}^{i-1}\binom{i-1}{k}\left(\frac{\theta\beta(\theta+1)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)}\right)^{k} \times\left(\frac{\gamma(\beta,\theta x)}{\Gamma(\beta)}\right)^{i-1-k}\right]$$
$$\times\left[\sum_{k=0}^{n-i}\binom{n-i}{k}\left(\frac{\Gamma(\beta)-\gamma(\beta,\theta x)}{\Gamma(\beta)}\right)^{k}\times\left(\frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x\right)e^{-\theta x}}{(1+\theta)}\right)^{n-i-k}\right]$$
$$\times\left[\frac{\theta^3\beta(1+x)e^{-\theta x}}{1+\theta} + \frac{\theta^\beta x^{\beta-1}e^{-\theta x}}{\Gamma(\beta)}\right]$$

2.5.2 Quantile function

Let $X \backsim MGL(\beta, \theta)$, the quantile function for the Mixed Gamma Lindley distribution is defined as the inverse of its CDF. Formally, for $0 < y < 1$,

$$x = F^{-1}(y),$$

we have:

$$y = \frac{1}{\theta\beta + 1}\left(\theta\beta - \frac{(\theta\beta + \theta^2\beta + \theta^2\beta x)\, e^{-\theta x}}{1+\theta} + \frac{\gamma(\beta, \theta x)}{\Gamma(\beta)}\right)$$

$$(\theta\beta + 1)\, y = \frac{\theta\beta(1+\theta)\Gamma(\beta) - (\theta\beta + \theta^2\beta + \theta^2\beta x)\Gamma(\beta)\, e^{-\theta x} + (\theta+1)\gamma(\beta, \theta x)}{(\theta+1)\Gamma(\beta)}$$

$$\Gamma(\beta)(\theta+1)(\theta\beta+1)\, y = \Gamma(\beta)\left(\theta\beta + \theta^2\beta\right) - \left(\theta\beta + \theta^2\beta + \theta^2\beta x\right)\Gamma(\beta)\, e^{-\theta x} + (\theta+1)\gamma(\beta, \theta x)$$

$$\Gamma(\beta)(\theta+1)(\theta\beta+1)\, y - \Gamma(\beta)\left(\theta\beta + \theta^2\beta\right) = (\theta+1)\gamma(\beta, \theta x) - \left(\theta\beta + \theta^2\beta + \theta^2\beta x\right)\Gamma(\beta)\, e^{-\theta x},$$

this equation can't be solved explicity, we need numerical methed to solve it.

2.6 Entropy of Mixed Gamma Lindley distribution

2.6.1 Renyi entropy

For a random variable follow the MGL distribution, the renyi entropy is formulated by substituting the PDF in Equation (2.7) into Equation (1.21):

$$R_\lambda(X) = \frac{1}{1-\lambda}\ln\left[\int_0^\infty \left(\frac{\theta^3\beta(1+x)\, e^{-\theta x}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta x^{\beta-1} e^{-\theta x}}{(\theta\beta+1)\Gamma(\beta)}\right)^\lambda\right] dx$$

$$= \frac{1}{1-\lambda}\ln\left[\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^\lambda \int_0^\infty (1+x)^\lambda e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^\lambda \int_0^\infty x^{(\beta-1)\lambda} e^{-\theta\lambda x} dx \end{array}\right]$$

$$= \frac{1}{1+\lambda} \ln \left[\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \int_0^\infty \sum_{i=0}^{\lambda} \binom{\lambda}{i} x^i 1^{\lambda-i} e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} \int_0^\infty x^{(\beta\lambda-\lambda)} e^{-\theta\lambda x} dx \end{array} \right]$$

$$= \frac{1}{1+\lambda} \ln \left[\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} \int_0^\infty x^i e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \int_0^\infty x^{(\lambda\beta-\lambda)} e^{-\theta\lambda x} dx \end{array} \right]$$

$$= \frac{1}{1+\lambda} \ln \left[\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} \frac{\Gamma(i+1)}{(\theta\lambda)^{i+1}} \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} \frac{\Gamma(\beta\lambda-\lambda+1)}{(\theta\lambda)^{(\beta\lambda-\lambda+1)}} \end{array} \right]$$

2.6.2 Tsallis Entropy

The Tsallis Entropy for the MGL distribution is derived using equations. (2.7) and (1.22) and is expressed as:

$$T_\lambda(X) = \frac{1}{\lambda-1}\left(1 - \int_0^\infty \left(\frac{\theta^3\beta(1+x)e^{-\theta x}}{(\theta\beta+1)(\theta+1)} + \frac{\theta^\beta x^{\beta-1} e^{-\theta x}}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} dx\right)$$

$$= \frac{1}{\lambda-1}\left(1 - \left(\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \int_0^\infty (1+x)^{\lambda} e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} \int_0^\infty x^{(\beta-1)\lambda} e^{-\theta\lambda x} dx \end{array} \right)\right)$$

$$= \frac{1}{\lambda-1}\left(1 - \left(\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \int_0^\infty \sum_{i=0}^{\lambda} \binom{\lambda}{i} x^i 1^{\lambda-i} e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} \int_0^\infty x^{(\beta\lambda-\lambda)} e^{-\theta\lambda x} dx \end{array} \right)\right)$$

$$= \frac{1}{\lambda-1}\left(1 - \left(\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} \int_0^\infty x^i e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \int_0^\infty x^{(\beta\lambda-\lambda)} e^{-\theta\lambda x} dx \end{array} \right)\right)$$

$$= \frac{1}{\lambda-1}\left(1 - \left(\begin{array}{c} \left(\frac{\theta^3\beta}{(\theta\beta+1)(\theta+1)}\right)^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} \frac{\Gamma(i+1)}{(\theta\lambda)^{i+1}} \\ + \left(\frac{\theta^\beta}{(\theta\beta+1)\Gamma(\beta)}\right)^{\lambda} \frac{\Gamma(\beta\lambda-\lambda+1)}{(\theta\lambda)^{(\beta\lambda-\lambda+1)}} \end{array} \right)\right)$$

2.7 Methods of estimation

2.7.1 Maximum likelihood Estimation

Consider a random sample $X_1, X_2, \cdots, X_n$ drawn from the MGL distribution. The likelihood function $L(x; \beta, \theta)$, which depends on parameters β and θ, is formulated as follows:

$$L(x_i; \beta, \theta) = \prod_{i=1}^{n} \left[\frac{\theta^3 \beta (1 + x_i) e^{-\theta x_i}}{(\theta\beta + 1)(\theta + 1)} + \frac{\theta^\beta x_i^{\beta-1} e^{-\theta x_i}}{(\theta\beta + 1)\Gamma(\beta)} \right]$$

$$= \prod_{i=1}^{n} \left(\frac{\left[\theta^3 \beta \Gamma(\beta)(1 + x_i) + \theta^\beta x_i^{\beta-1}(\theta + 1)\right] e^{-\theta x_i}}{(\theta\beta + 1)(\theta + 1)\Gamma(\beta)} \right)$$

$$= \frac{e^{-\sum_{i=1}^{n} \theta x_i}}{(\theta\beta + 1)^n (\theta + 1)^n (\Gamma(\beta))^n} \prod_{i=1}^{n} \left(\theta^3 \beta \Gamma(\beta)(1 + x_i) + \theta^\beta x_i^{\beta-1}(\theta + 1) \right)$$

The ln $-$ likelihood function is

$$l = \ln(L(x_i; \beta, \theta)) = -\sum_{i=1}^{n} \theta x_i - n\left[\ln(\theta\beta + 1) + \ln(\theta + 1) + \ln(\Gamma(\beta))\right]$$
$$+ \sum_{i=1}^{n} \ln\left(\theta^3 \beta \Gamma(\beta)(1 + x_i) + \theta^\beta x_i^{\beta-1}(\theta + 1)\right)$$

Next, we compute the partial derivatives with respect to θ and β, yielding :

$$\frac{\partial l}{\partial \theta} = -\sum_{i=1}^{n} x_i - n\left[\frac{\beta}{\theta\beta + 1} + \frac{1}{\theta + 1}\right]$$
$$+ \sum_{i=1}^{n} \left(\frac{3\theta^2 \beta \Gamma(\beta)(1 + x_i) + x_i^{\beta-1}(\theta^\beta + \beta\theta^{\beta-1}(\theta + 1)}{\theta^3 \beta \Gamma(\beta)(1 + x_i) + \theta^\beta x_i^{\beta-1}(\theta + 1)} \right)$$

$$\frac{\partial l}{\partial \theta} = 0$$

$$\sum_{i=1}^{n} \left(\frac{3\theta^2 \beta \Gamma(\beta)(1 + x_i) + x_i^{\beta-1}(\theta^\beta + \beta\theta^{\beta-1}(\theta + 1)}{\theta^3 \beta \Gamma(\beta)(1 + x_i) + \theta^\beta x_i^{\beta-1}(\theta + 1)} \right) - \sum_{i=1}^{n} x_i - n\left[\frac{\beta}{\theta\beta + 1} + \frac{1}{\theta + 1}\right] = 0$$

$$\frac{\partial l}{\partial \beta} = -n\left[\frac{\theta}{\theta\beta + 1} + \frac{\acute{\Gamma}(\beta)}{\Gamma(\beta)}\right]$$

$$+\sum_{i=1}^{n}\left(\frac{\theta^3(1+x_i)\left(\Gamma(\beta)+\beta\acute{\Gamma}(\beta)\right)+(\theta+1)\left[\ln\theta\times\theta^\beta x_i^{\beta-1}+\theta^\beta\ln x_i\times x_i^{\beta-1}\right]}{\theta^3\beta\Gamma(\beta)(1+x_i)+\theta^\beta x_i^{\beta-1}(\theta+1)}\right)$$

$$\frac{\partial l}{\partial\beta}=0$$

$$\sum_{i=1}^{n}\left(\frac{\theta^3(1+x_i)\left(\Gamma(\beta)+\beta\acute{\Gamma}(\beta)\right)+(\theta+1)\left[\ln\theta\times\theta^\beta x_i^{\beta-1}+\theta^\beta\ln x_i\times x_i^{\beta-1}\right]}{\theta^3\beta\Gamma(\beta)(1+x_i)+\theta^\beta x_i^{\beta-1}(\theta+1)}\right)$$

$$-n\left[\frac{\theta}{\theta\beta+1}+\frac{\acute{\Gamma}(\beta)}{\Gamma(\beta)}\right]=0$$

The maximum liklihood estimates for (θ,β) can be derived by solving a system of non linear equation.

2.7.2 Ordinary Least Squares estimation

In the context of a random sample of size n drawn from the MGL distribution, let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ represent the ordered sample observations. The ordinary least squares (OLS) estimators for the parameters θ and β can be derived by minimizing the following exppression:

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2$$

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[\frac{\theta\beta}{\theta\beta+1}-\frac{(\theta\beta+\theta^2\beta+\theta^2\beta x_i)e^{-\theta x_i}}{(\theta\beta+1)(\theta+1)}+\frac{\gamma(\beta,\theta x_i)}{(\theta\beta+1)\Gamma(\beta)}-\frac{i}{n+1}\right]^2$$

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[\begin{array}{c}\frac{\theta\beta(\theta+1)\Gamma(\beta)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}+(\theta+1)\gamma(\beta,\theta x_i)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\\-\frac{i}{n+1}\end{array}\right]^2$$

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{\theta\beta(\theta+1)\Gamma(\beta)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}+(\theta+1)\gamma(\beta,\theta x_i)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\right)^2\\-2\left(\frac{\theta\beta(\theta+1)\Gamma(\beta)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}+(\theta+1)\gamma(\beta,\theta x_i)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\right)\times\left(\frac{i}{n+1}\right)\\+\left(\frac{i}{n+1}\right)^2\end{array}\right]$$

Derivate $T(x;\beta,\theta)$ with respect to β and θ to find the estimators $\hat{\beta}$ and $\hat{\theta}$.

Since there is no closed form for these equations, then the ordinary least squares, can be solved numerically.

2.7.3 Weighted Least Squares estimation

The weighted least squares (WLS) estimator for parameters θ and β in the mixted Gamma Lindley distribution is derived by minimizing the following objective function:

$$W(x;\beta,\theta)=\sum_{i=1}^{n}w_i\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2$$

$$=\sum_{i=1}^{n}\frac{(n+1)^2(n+1)}{i(n-i+1)}\left[\begin{array}{c}\frac{\theta\beta}{\theta\beta+1}-\frac{\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}}{(\theta\beta+1)(\theta+1)}+\frac{\gamma(\beta,\theta x_i)}{(\theta\beta+1)\Gamma(\beta)}\\-\frac{i}{n+1}\end{array}\right]^2$$

$$=\sum_{i=1}^{n}\frac{(n+1)^2(n+1)}{i(n-i+1)}\left[\begin{array}{c}\left(\frac{\theta\beta(\theta+1)\Gamma(\beta)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}+(\theta+1)\gamma(\beta,\theta x_i)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\right)^2\\-2\left(\frac{\theta\beta(\theta+1)\Gamma(\beta)-\left(\theta\beta+\theta^2\beta+\theta^2\beta x_i\right)e^{-\theta x_i}+(\theta+1)\gamma(\beta,\theta x_i)}{(\theta\beta+1)(\theta+1)\Gamma(\beta)}\right)\times\left(\frac{i}{n+1}\right)\\+\left(\frac{i}{n+1}\right)^2\end{array}\right]$$

Similarly, the weighted least squares estimates of $\Theta=(\beta,\theta)$ can be obtained by solving the system $\left\{\frac{\partial W}{\partial\beta}=0,\frac{\partial W}{\partial\theta}=0\right\}$. This system has no exact solution, we can use a numerical method to solve it.

2.8 Application to real data

The mixed Gamma Lindley distribution has been fitted to some real life time data set and it gives better fit than transmuted Shanker[33], Lindley [34], Ishita [35] and Weibull [36] distribution.

To compare transmued Shanker, Lindley, Ishita, Weibull to the suggested MGL distribution, the ln −likelihood (LL), Akaike Information Criterion (AIC), Corrected Akaike Information Criterion $(CAIC)$, Bayesian Information Criterion (BIC), Hannan-Quin Information Criterion $(HQIC)$ and Kolmogorov-Smirnov statistics (KS) are computed using the R program, for real lifetime data set in the table below:

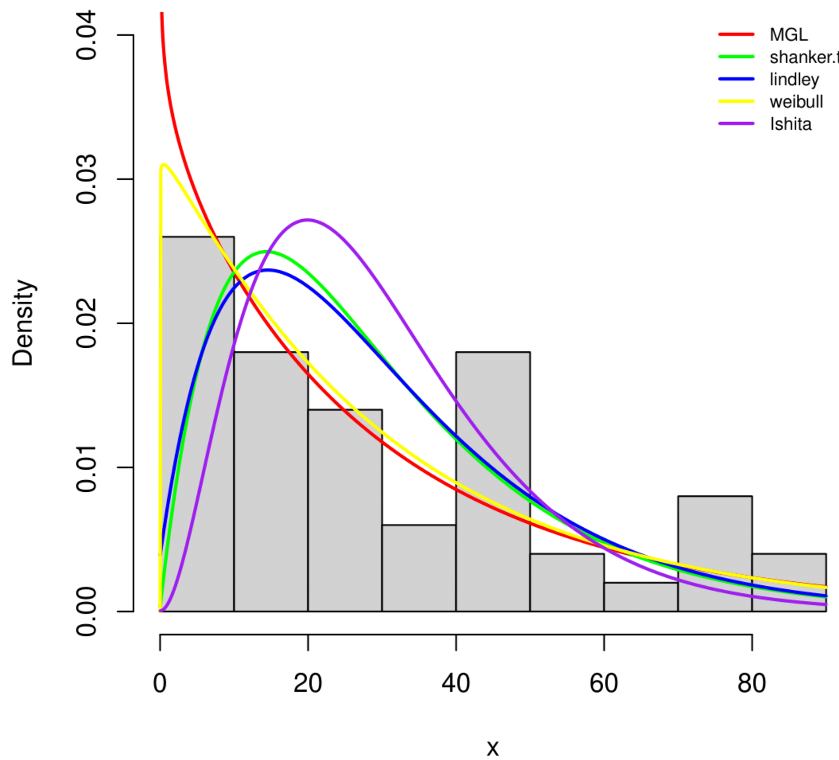
0,12	0 43	0 92	1,14	1,24	1,61	1,93	2,38	4,51	5,09	6,79
7,64	8,45	11,90	11,94	13,01	13,25	14,32	17,47	18,10	18,66	19,23
24,39	25,01	26,41	26,80	27,75	29,69	29,84	31,65	32,64	35,00	40,70
42,34	43,05	43,40	44,36	45,40	48,14	49,10	49,44	51,17	58,62	60,29
72,13	72,22	72,25	72,29	85,20	89,52					

The best distribution among the five distribution which gives a good fit to lifetime data corresponds to the lowest negative LL, AIC, BIC, $HQIC$ and KS statistics and highest p-value.

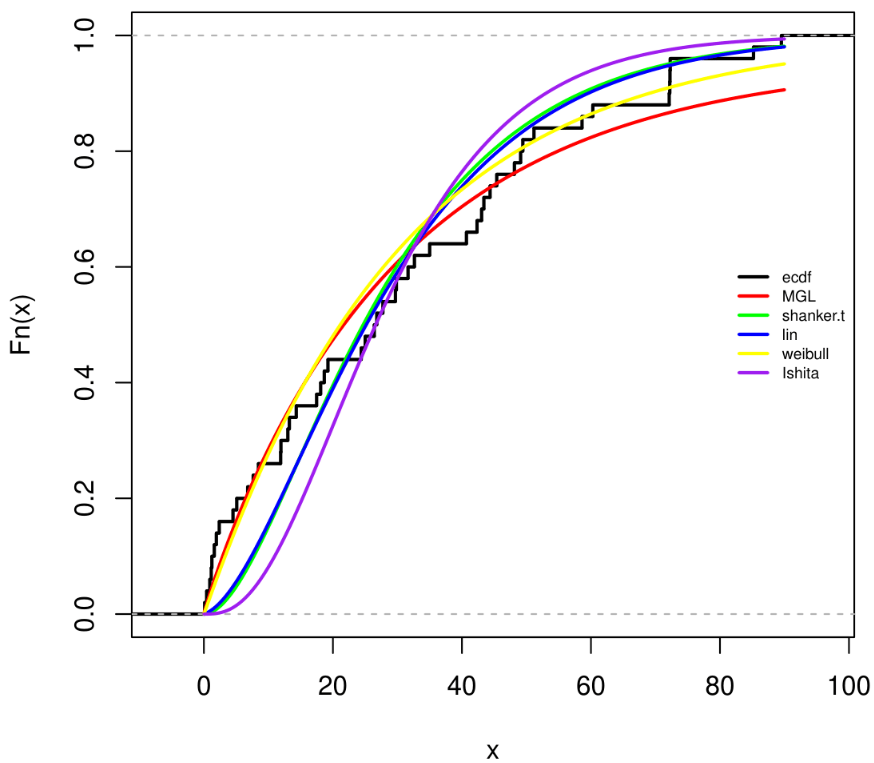
Table 2: $-LL$, AIC, $CAIC$, BIC, $HQIC$, KS *statistic and the* $p-values$ *of the fitted distributions.*

Distribution	$-LL$	AIC	$CAIC$	BIC	$HQIC$	KS	$p-value$
MGL	**220,261**	**444,522**	**444,777**	**444,346**	**445,978**	**0,098**	**0,678**
Shanker.t	231,595	467,191	467,446	471,015	468,647	0,151	0,183
Lindlely	226,974	455,948	456,032	457,860	456,676	0,142	0,237
Weibull	220,349	444,698	444,953	444,953	446,154	0,111	0,529
Ishita	250,2476	502,4952	502,785	504,4072	503,2233	0,20512	0,025

From table (2) we can observe that MGL distribution is flexible and appropriate for the fiiting of the data set than the computing models because it has the least $-LL$, AIC, $CAIC$, BIC, $HQIC$ and KS statistics and the highest p-value $p = 0,678$.
We can evaluate the effectiveness of new model by considering two illustrations for the data set which show the histogram of data, and the fitted distributions as it is shown in the figures below.



Histogram of data and pdf's of the fitted distributions



The empirical distribution function and the cdfs of the competating models

Chapter 3

Weighted Mixed Gamma Lindley with some statistical features

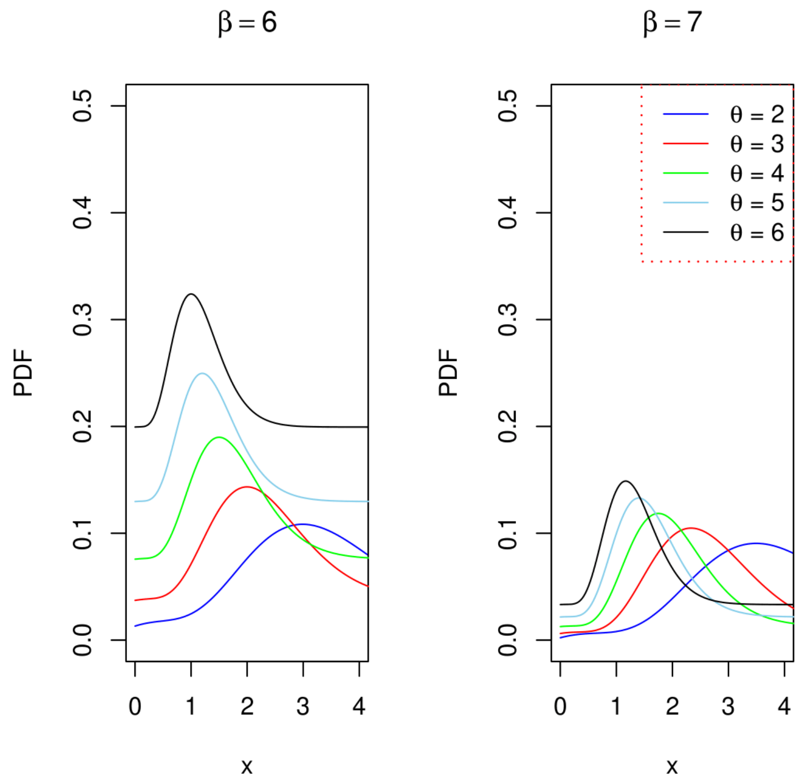
3.1 The suggested distribution

This section introduces the probability density function (pdf) and cumulative distribution function (cdf) of weighted MGL distribution, the new model is defined by putting $w(x) = x$ in 1.1 and using the pdf of MGL distribution which gives:

$$g(x;\beta,\theta) = \frac{\theta^4\beta\left(x+x^2\right)e^{-\theta x}}{\theta\beta(\theta+2)+\beta(\theta+1)} + \frac{\theta^{\beta+1}(\theta+1)\,x^\beta e^{-\theta x}}{\left(\theta\beta(\theta+2)+\beta(\theta+1)\right)\Gamma(\beta)}$$

$$= \frac{\theta^4\left(x+x^2\right)e^{-\theta x}}{\left(\theta^2+3\theta+1\right)} + \frac{\theta^{\beta+1}(\theta+1)\,x^\beta e^{-\theta x}}{\beta\Gamma(\beta)\left(\theta^2+3\theta+1\right)}$$

$$= \frac{\theta^4\Gamma(\beta+1)\left(x+x^2\right)+\theta^{\beta+1}(\theta+1)\,x^\beta}{\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)}e^{-\theta x} \tag{3.1}$$



The pdf of WMGLD for different values of β and θ.

The cumulative distribution function (cdf) is derived as follows:

$$G(x;\beta,\theta) = P(X \leq x) = \int_0^x g(t;\beta,\theta)\,dt$$

$$= \int_0^x \left(\frac{\theta^4 (t+t^2) e^{-\theta t}}{\theta^2+3\theta+1} + \frac{\theta^{\beta+1}(\theta+1) t^{\beta} e^{-\theta t}}{(\theta^2+3\theta+1)\Gamma(\beta+1)} \right) dt$$

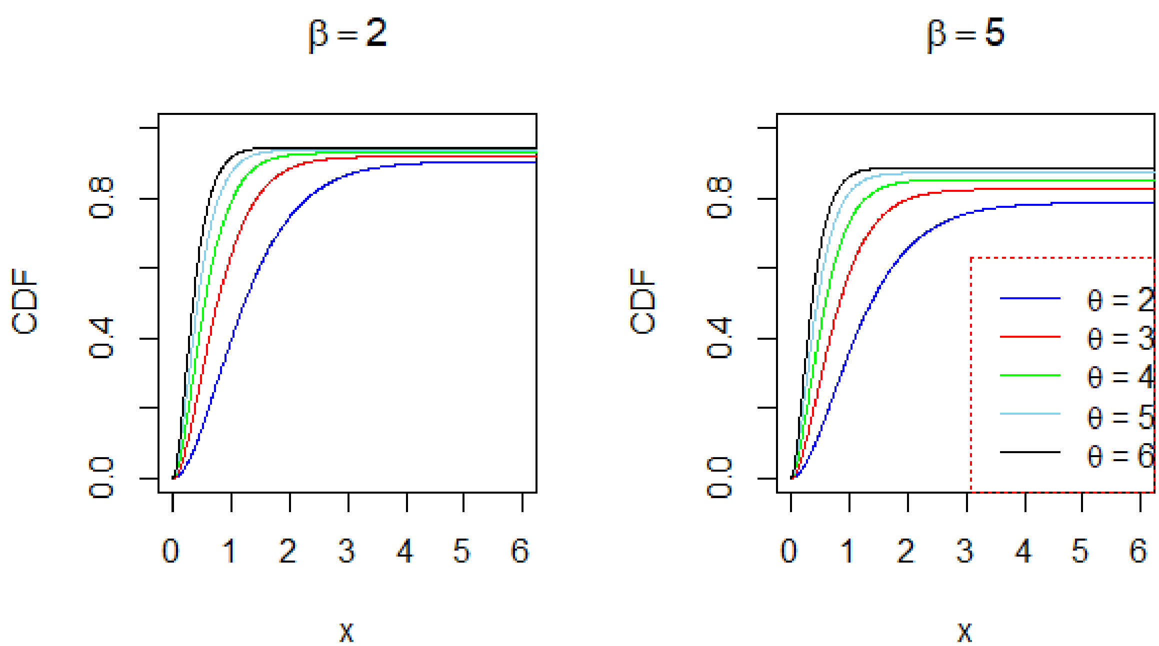
$$= \frac{\theta^4}{\theta^2+3\theta+1} \int_0^x (t+t^2) e^{-\theta t} dt$$

$$+ \frac{\theta^{\beta+1}(\theta+1)}{(\theta^2+3\theta+1)\Gamma(\beta+1)} \int_0^x t^{\beta} e^{-\theta t} dt$$

$$= \frac{\theta^4}{\theta^2+3\theta+1} \left(\int_0^x t e^{-\theta t} dt + \int_0^x t^2 e^{-\theta t} dt \right)$$

$$+ \frac{\theta^{\beta+1}(\theta+1)}{(\theta^2+3\theta+1)\Gamma(\beta+1)} \int_0^x t^{\beta} e^{-\theta t} dt$$

$$= \frac{\theta^4}{\theta^2+3\theta+1}\left(\frac{\theta+2}{\theta^3} - e^{-\theta x}\left(\frac{x^2}{\theta} + \frac{x(\theta+2)}{\theta^2} + \frac{\theta+2}{\theta^3}\right)\right)$$
$$+\frac{\theta^{\beta+1}(\theta+1)}{(\theta^2+3\theta+1)\,\Gamma(\beta+1)}\frac{\gamma(\beta+1,\theta x)}{\theta^{\beta+1}}$$
$$= \frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)+(\theta+1)\,\gamma(\beta+1,\theta x)}{(\theta^2+3\theta+1)\,\Gamma(\beta+1)} \tag{3.2}$$



The cdf of WMGLD for different values of θ and β.

3.2 Statistical features

In this section, key statistical properties of the Weighted Gamma Lindley distribution are presented, such as the r^{th} moment, mean, variance, coefficients of variation, skewness, and kurtosis.

3.2.1 Moments of weighted Mixed Gamma Lindley distribution

let X be a random variable following the WMGL distribution with parameters θ and β. The r^{th} order moment $E\left(X^r\right)$ is derived as:

$$\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \left[\frac{\theta^4\Gamma\left(\beta+1\right)\left(x+x^2\right)e^{-\theta x}+\theta^{\beta+1}\left(\theta+1\right)x^\beta e^{-\theta x}}{\Gamma\left(\beta+1\right)\left(\theta^2+3\theta+1\right)}\right]dx \\
&= \int_0^\infty x^r \left(\frac{\theta^4\left(x+x^2\right)e^{-\theta x}}{\theta^2+3\theta+1}\right)dx + \int_0^\infty x^r\left(\frac{\theta^{\beta+1}\left(\theta+1\right)x^\beta e^{-\theta x}}{\Gamma\left(\beta+1\right)\left(\theta^2+3\theta+1\right)}\right)dx \\
&= \frac{\theta^4}{\theta^2+3\theta+1}\left(\int_0^\infty x^{r+1}e^{-\theta x}dx+\int_0^\infty x^{r+2}e^{-\theta x}dx\right) \\
&+ \frac{\theta^{\beta+1}\left(\theta+1\right)}{\Gamma\left(\beta+1\right)\left(\theta^2+3\theta+1\right)}\int_0^\infty x^{r+\beta}e^{-\theta x}dx
\end{aligned}$$

substituting the probability density function $g\left(x;\beta,\theta\right)$, we obtain:

$$E\left(X^r\right) = \frac{\theta^4}{\theta^2+3\theta+1}\left[\frac{\Gamma\left(r+2\right)}{\theta^{r+2}}+\frac{\Gamma\left(r+3\right)}{\theta^{r+3}}\right]+\frac{\left(\theta+1\right)\Gamma\left(r+\beta+1\right)}{\Gamma\left(\beta+1\right)\left(\theta^2+3\theta+1\right)\theta^r} \tag{3.3}$$

Using this result, the first four moments are computed as follows:

1.First moment (Mean):

$$\begin{aligned}
E\left(X\right) &= \frac{2\theta+6}{\theta^2+3\theta+1}+\frac{\left(\theta+1\right)\left(\beta+1\right)}{\left(\theta^2+3\theta+1\right)\theta} \\
&= \frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}
\end{aligned} \tag{3.4}$$

2.Second moment:

$$\begin{aligned}
E\left(X^2\right) &= \frac{6\theta+24}{(\theta^2+3\theta+1)\theta}+\frac{\left(\beta+1\right)\left(\beta+2\right)\left(\theta+1\right)}{\left(\theta^2+3\theta+1\right)\theta^2} \\
&= \frac{6\theta^2+24\theta+\left(\beta^2+3\beta+2\right)\left(\theta+1\right)}{\theta^4+3\theta^3+\theta^2}
\end{aligned} \tag{3.5}$$

3.Third moment:

$$E\left(X^3\right) = \frac{24\theta + 120}{(\theta^2 + 3\theta + 1)\,\theta^2} + \frac{(\beta+1)(\beta+2)(\beta+3)(\theta+1)}{(\theta^2 + 3\theta + 1)\,\theta^3}$$
$$= \frac{24\theta^2 + 120\theta + (\beta^3 + 6\beta^2 + 11\beta + 6)(\theta+1)}{\theta^5 + 3\theta^4 + \theta^3} \tag{3.6}$$

4.Fourth moment:

$$E\left(X^4\right) = \frac{120\theta + 720}{(\theta^2 + 3\theta + 1)\,\theta^3} + \frac{(\theta+1)(\beta+4)(\beta+3)(\beta+2)(\beta+1)}{(\theta^2 + 3\theta + 1)\,\theta^4}$$
$$= \frac{120\theta^2 + 720\theta + (\theta+1)(\beta^4 + 10\beta^3 + 35\beta^2 + 50\beta + 24)}{\theta^6 + 3\theta^5 + \theta^4} \tag{3.7}$$

Using these moments, the variance $V(X) = \sigma_X^2$ is defined by combining equations (1.4), (3.4) and (3.5)

$$V(X) = \frac{6\theta^2 + 24\theta + (\beta^2 + 3\beta + 2)(\theta+1)}{\theta^4 + 3\theta^3 + \theta^2} - \left(\frac{2\theta^2 + 7\theta + \theta\beta + \beta + 1}{\theta^3 + 3\theta^2 + \theta}\right)^2$$

As a result, the WMGLdistribution standard deviation

$$\sigma_X = \sqrt{\frac{6\theta^2 + 24\theta + (\beta^2 + 3\beta + 2)(\theta+1)}{\theta^4 + 3\theta^3 + \theta^2} - \left(\frac{2\theta^2 + 7\theta + \theta\beta + \beta + 1}{\theta^3 + 3\theta^2 + \theta}\right)^2} \tag{3.8}$$

The coefficient of variation is a

$$CV = \frac{\sqrt{\frac{6\theta^2+24\theta+(\beta^2+3\beta+2)(\theta+1)}{\theta^4+3\theta^3+\theta^2} - \left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)^2}}{\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}}$$
$$= \frac{\sqrt{(\theta^2 + 3\theta + 1)\left[6\theta^2 + 24\theta + (\beta^2 + 3\beta + 2)(\theta+1)\right] - (2\theta^2 + 7\theta + \theta\beta + \beta + 1)^2}}{2\theta^2 + 7\theta + \theta\beta + \beta + 1}$$

3.2.2 Moment generating function of WMGL

The moment generating function for the weighted mixing Gamma Lindley distribution is obtained by integrating the expressions equations (1.3) and (3.1). The derivation proceeds as follows:

$$\begin{aligned}
M_X(t) &= \int_0^\infty e^{tx}\left(\frac{\theta^4\Gamma(\beta+1)\left(x+x^2\right)e^{-\theta x}+\theta^{\beta+1}(\theta+1)\,x^{\beta}e^{-\theta x}}{\Gamma(\beta+1)\left(\theta^2+3\theta+1\right)}\right)dx \\
&= \frac{\theta^4}{\theta^2+3\theta+1}\int_0^\infty \left(x+x^2\right)e^{-(\theta-t)x}dx + \frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)\left(\theta^2+3\theta+1\right)}\int_0^\infty x^{\beta}e^{-(\theta-t)x}dx \\
&= \frac{\theta^4}{\theta^2+3\theta+1}\left(\int_0^\infty xe^{-(\theta-t)x}dx+\int_0^\infty x^2e^{-(\theta-t)x}dx\right) \\
&+ \frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)\left(\theta^2+3\theta+1\right)}\int_0^\infty x^{\beta}e^{-(\theta-t)x}dx \\
&= \frac{\theta^4}{\theta^2+3\theta+1}\left(\frac{\Gamma(2)}{(\theta-t)^2}+\frac{\Gamma(3)}{(\theta-t)^3}\right)+\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)\left(\theta^2+3\theta+1\right)}\frac{\Gamma(\beta+1)}{(\theta-t)^{(\beta+1)}} \\
&= \frac{\theta^4}{\theta^2+3\theta+1}\left(\frac{\theta-t+2}{(\theta-t)^3}\right)+\frac{\theta^{\beta+1}(\theta+1)}{\left(\theta^2+3\theta+1\right)(\theta-t)^{(\beta+1)}}
\end{aligned}$$

3.2.3 Skewness

The definition of skewness employs equations (1.6) , (3.4) , (3.5) , (3.6) , and (3.8) as follows:

$$SK(X)=\frac{\left(\frac{24\theta^2+120\theta+\left(\beta^3+6\beta^2+11\beta+6\right)(\theta+1)}{\theta^5+3\theta^4+\theta^3}\right)+2\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)^3 -3\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)\left(\frac{6\theta^2+24\theta+\left(\beta^2+3\beta+2\right)(\theta+1)}{\theta^4+3\theta^3+\theta}\right)}{\left(\frac{6\theta^2+24\theta+(\beta^2+3\beta+2)(\theta+1)}{\theta^4+3\theta^3+\theta^2}-\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)^2\right)^{\frac{3}{2}}}$$

3.2.4 Kurtosis

Using equations (1.8) , (3.4) , (3.5) , (3.6) , (3.7), and (3.8) , kurtosis is defined as:

$$Kur(X)=\frac{\left(\frac{120\theta^2+720\theta+(\theta+1)\left(\beta^4+10\beta^3+35\beta^2+50\beta+24\right)}{\theta^6+3\theta^5+\theta^4}\right)+6\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)^2 -4\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)\left(\frac{24\theta^2+120\theta+\left(\beta^3+6\beta^2+11\beta+6\right)(\theta+1)}{\theta^5+3\theta^4+\theta^3}\right) +\left(\frac{6\theta^2+24\theta+\left(\beta^2+3\beta+2\right)(\theta+1)}{\theta^4+3\theta^3+\theta^2}\right)-3\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta}\right)^4}{\left(\frac{6\theta^2+24\theta+(\beta^2+3\beta+2)(\theta+1)}{\theta^4+3\theta^3+\theta^2}-\left(\frac{2\theta^2+7\theta+\theta\beta+\beta+1}{\theta^3+3\theta^2+\theta^2}\right)^2\right)^2}$$

Table 3: Mean, standard deviation, skewness, kurtosis, and C.V of variation for WMGL distribution for different values of θ and β.

β	θ	μ	σ	$skewness$	$kurtosis$	$C.V\%$
$0,5$	$0,45$	$4,5968$	$3,5545$	$1,4142$	$2,8244$	$77,3254$
$0,5$	$0,65$	$3,2936$	$2,4944$	$1,3751$	$2,6746$	$75,7343$
$0,5$	$0,85$	$2,5663$	$1,9168$	$1,3555$	$2,6076$	$74,691$
1	$0,05$	$41,7354$	$29,4347$	$1,4043$	$2,9471$	$70,5271$
1	$0,25$	$9,1034$	$6,2936$	$1,3454$	$2,6678$	$69,1348$
1	$0,45$	$5,228$	$3,57$	$1,3162$	$2,5441$	$68,2866$
1	$0,65$	$3,67$	$2,4913$	$1,3036$	$2,4931$	$67,8846$
1	$0,85$	$2,8211$	$1,9106$	$1,2989$	$2,4742$	$67,7273$
$1,5$	$0,05$	$50,846$	$32,0171$	$1,2562$	$2,3644$	$62,9689$
$1,5$	$0,25$	$10,4828$	$6,5528$	$1,2365$	$2,284$	$62,5104$
$1,5$	$0,45$	$5,8592$	$3,669$	$1,2301$	$2,2565$	$62,6194$
$1,5$	$0,65$	$4,0463$	$2,547$	$1,2296$	$2,2512$	$62,9465$
$1,5$	$0\ 85$	$3,0758$	$1,9485$	$1,2321$	$2,257$	$63,35$

From the results in the table, we can see that as θ increases, the mean and the standard deviation, skewness, kurtosis and C.V decrease, and when β increases, the values of μ and σ increase as the skewness, kurtosis, the C.V decrease.

3.3 Reliability analysis

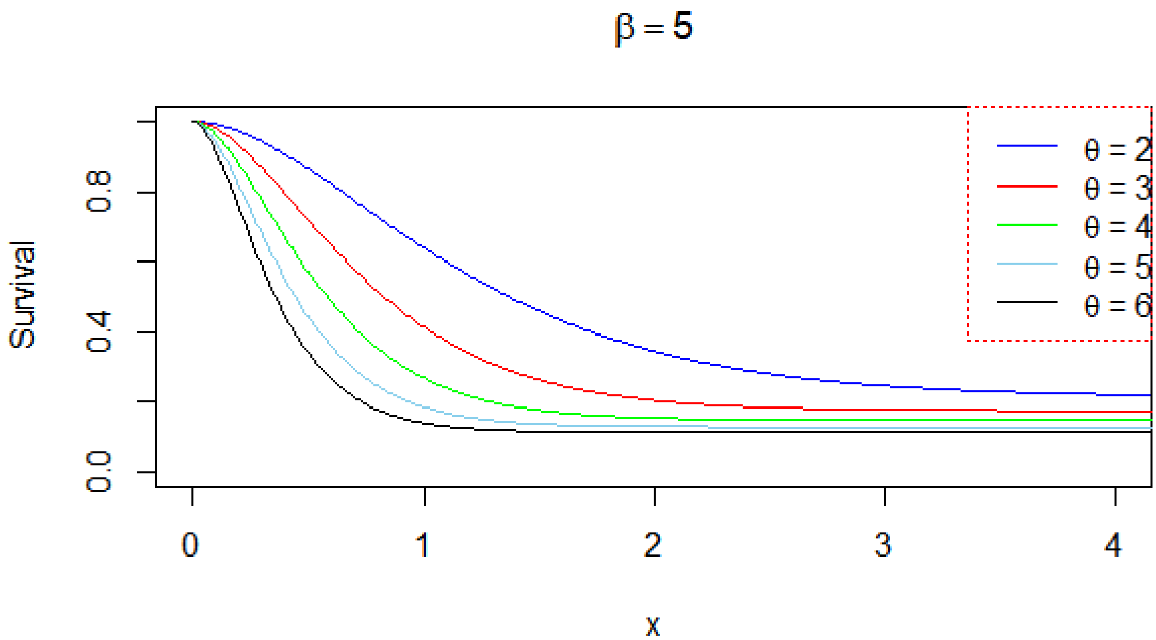
This section presents the Reliability related function of the weighted mixed Gamma Lindley distribution, including the survival function, hazard rate function, reverse hazard function, cumulative hazard function, and odds rate function.

3.3.1 Survival function

The survival (or reliability) function corresponding to the weighted mixed Gamma Lindley distriibution is formulated through the following equations (1.12) and (3.2) :

$$S\left(t\right)=1-\left(\frac{\Gamma\left(\beta+1\right)\left(\theta^{2}+2\theta-\left(\theta^{3}t^{2}+\theta^{3}t+2\theta^{2}t+\theta^{2}+2\theta\right)e^{-\theta t}\right)+\left(\theta+1\right)\gamma\left(\beta+1,\theta t\right)}{\left(\theta^{2}+3\theta+1\right)\Gamma\left(\beta+1\right)}\right)$$

$$= \frac{(\theta^2+3\theta+1)\,\Gamma(\beta+1) - \left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta)\,e^{-\theta t}\right)\\ +(\theta+1)\,\gamma(\beta+1,\theta t)\end{array}\right)}{(\theta^2+3\theta+1)\,\Gamma(\beta+1)} \tag{3.9}$$



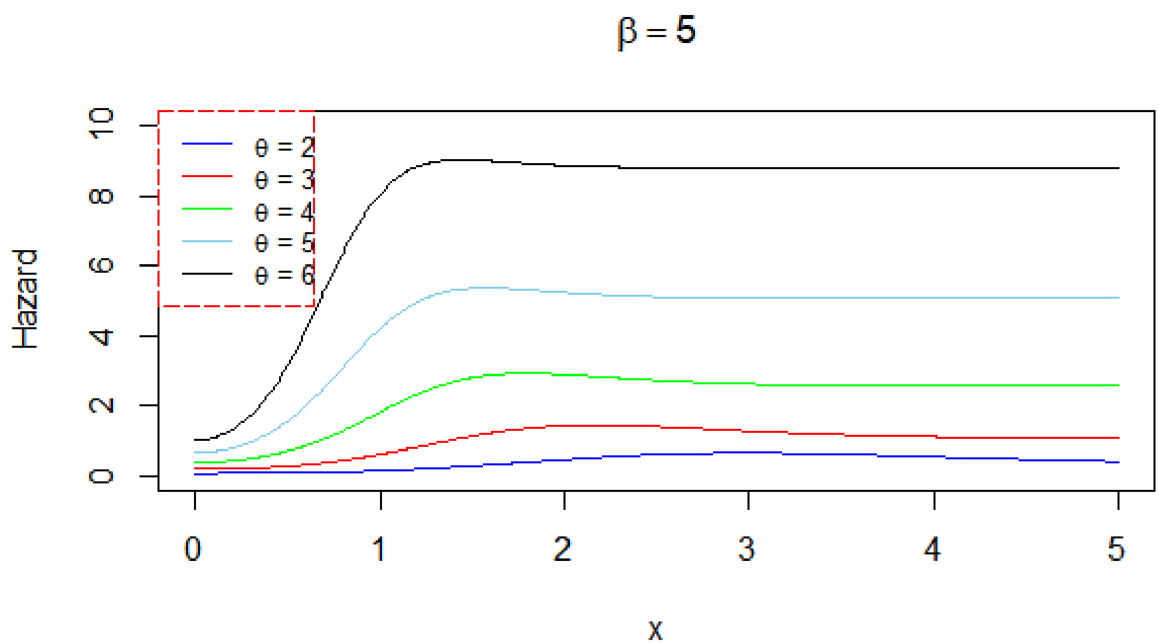
The survival function of WMGLD for different values of θ and β.

3.3.2 Hazard rate function

The hazard rate function $h(t)$ is derived by substituting the probability density function $g(t,\beta,\theta)$ and the survival function $S(t)$ into equation (1.13) :

$$h(t) = \frac{\frac{\theta^4\Gamma(\beta+1)\left(t+t^2\right)+\theta^{\beta+1}(\theta+1)t^\beta}{(\theta^2+3\theta+1)\Gamma(\beta+1)}e^{-\theta t}}{\frac{(\theta^2+3\theta+1)\Gamma(\beta+1)-\left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta)\,e^{-\theta t}\right)\\ +(\theta+1)\,\gamma(\beta+1,\theta t)\end{array}\right)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}}$$

$$= \frac{\theta^4\Gamma(\beta+1)\,(t+t^2)\,e^{-\theta t}+\theta^{\beta+1}(\theta+1)\,e^{-\theta t}}{(\theta^2+3\theta+1)\,\Gamma(\beta+1) - \left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta)\,e^{-\theta t}\right)\\ +(\theta+1)\,\gamma(\beta+1,\theta t)\end{array}\right)}$$



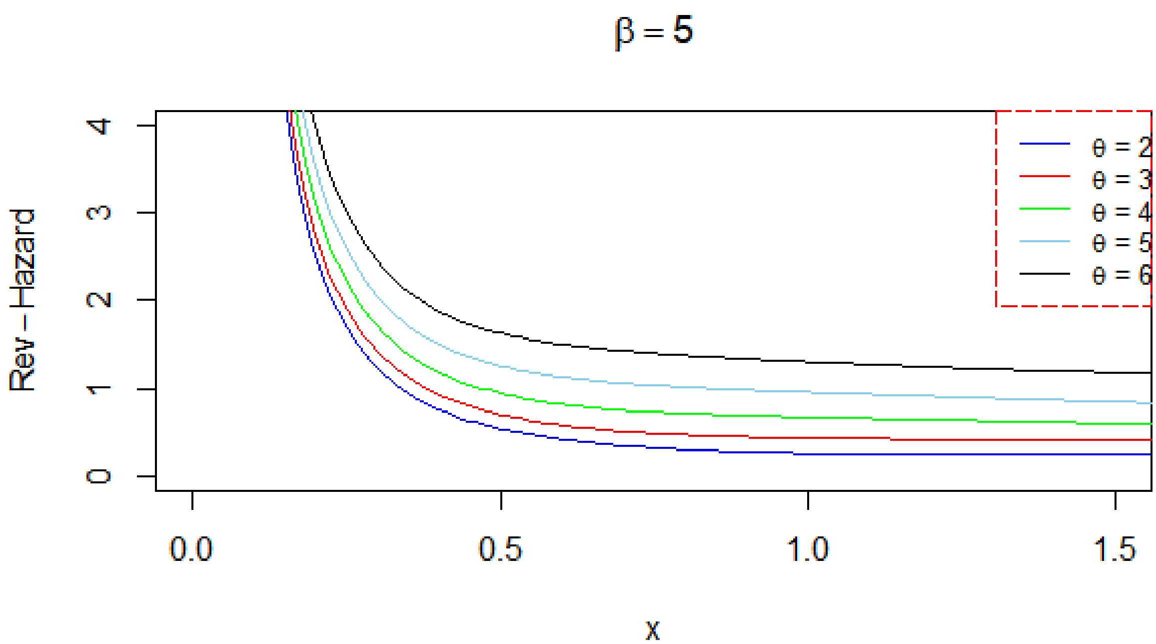
The hazard rate function of WMGLD for different values of θ and β.

3.3.3 Reversed hazard function

The reversed hazard rate of weighted Mixed Gamma Lindlely distribution can be derived using equation (1.16) , (3.1) and (3.2) as follows:

$$rh\,(t) = \frac{\frac{\theta^4\Gamma(\beta+1)\left(t+t^2\right)e^{-\theta t}+\theta^{\beta+1}(\theta+1)t^{\beta}e^{-\theta t}}{(\theta^2+3\theta+1)\Gamma(\beta+1)}}{\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-(\theta^3x^2+\theta^3t+2\theta^2t+\theta^2+2\theta)e^{-\theta t}\right)+(\theta+1)\gamma(\beta+1,\theta t)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}}$$

$$= \frac{\theta^4\Gamma\left(\beta+1\right)\left(t+t^2\right)e^{-\theta t}+\theta^{\beta+1}\left(\theta+1\right)t^{\beta}e^{-\theta t}}{\Gamma\left(\beta+1\right)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)+\left(\theta+1\right)\gamma\left(\beta+1,\theta t\right)}$$



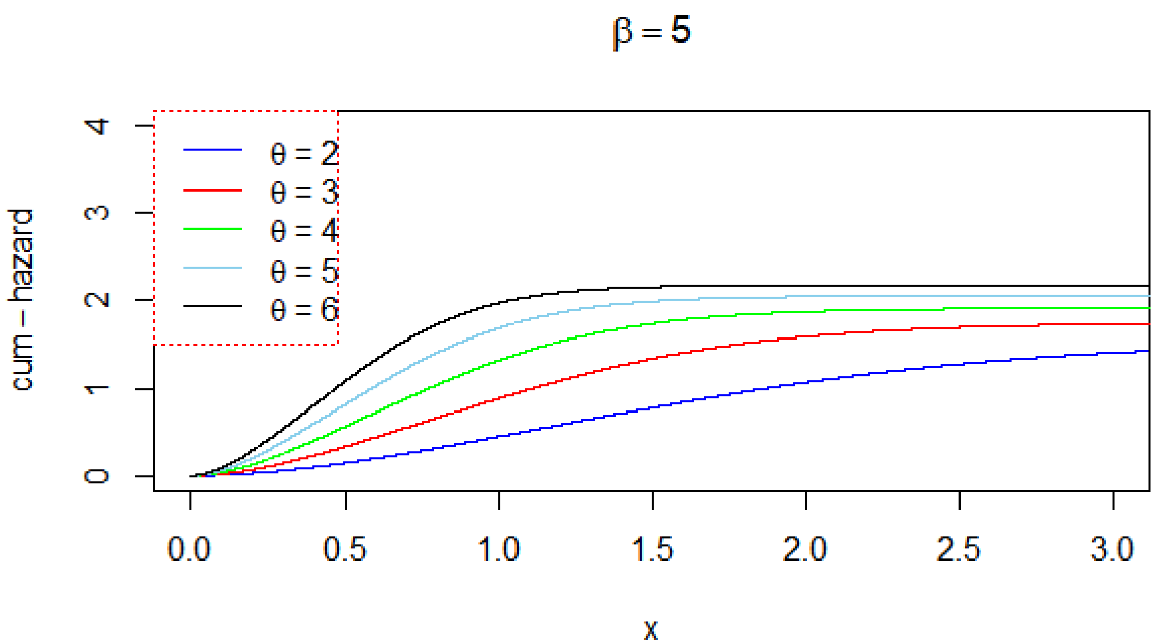
The reversed hazard rate function of MGLD for different values of θ and β.

3.3.4 Cumulative hazard function

Using equation.(1.15) and (3.9), the cumulative hazard function $H(t)$ for the weighted MGL distribution is derived as follows:

$$H(t) = -\ln\left[\frac{\begin{array}{c}(\theta^2+3\theta+1)\,\Gamma(\beta+1)\\ -\left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta)\,e^{-\theta t}\right)\\ +(\theta+1)\,\gamma(\beta+1,\theta t)\end{array}\right)\end{array}}{(\theta^2+3\theta+1)\,\Gamma(\beta+1)}\right]$$

$$= -\ln\left(\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)+(\theta+1)\,\gamma(\beta+1,\theta t)\right) + \ln\left(\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)\right)$$



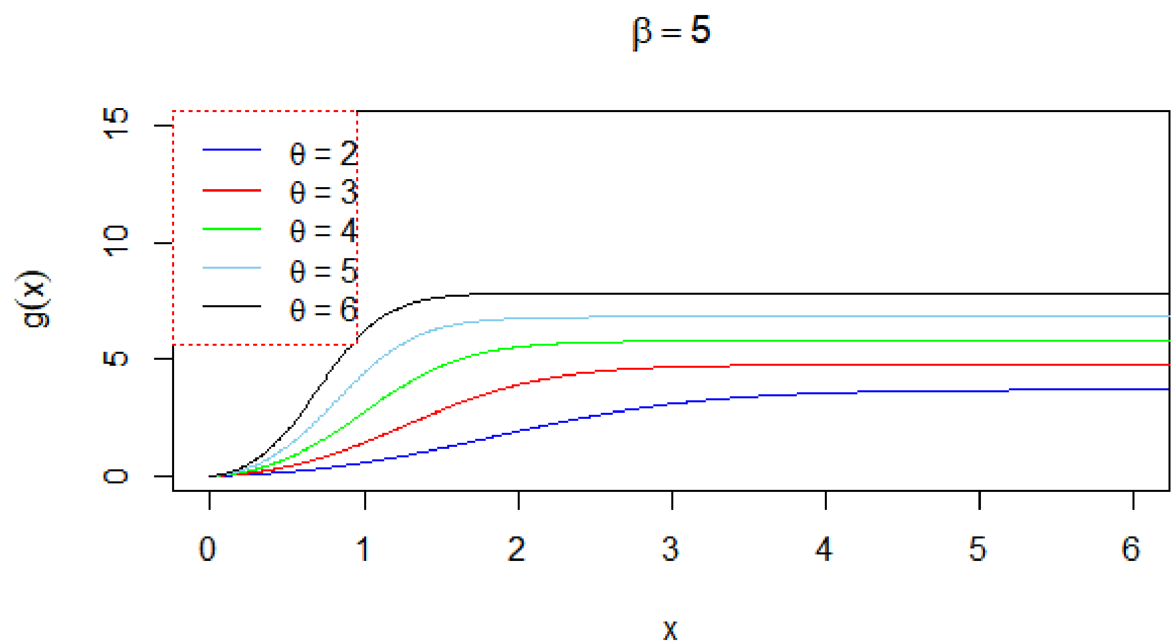
The cumulative hazard rate function of WMGLD for different values of θ and β.

3.3.5 Odds rate function

The odds rate function for the weighted MGL distribution can be derived by substituting equations (3.2) and (3.9) into the general formula provided in (1.17). This yields the following expression:

$$O(t) = \frac{\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)+(\theta+1)\gamma(\beta+1,\theta t)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}}{\frac{(\theta^2+3\theta+1)\Gamma(\beta+1)-\left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)\\+(\theta+1)\gamma(\beta+1,\theta t)\end{array}\right)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}}$$

$$= \frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)+(\theta+1)\gamma(\beta+1,\theta t)}{(\theta^2+3\theta+1)\Gamma(\beta+1)-\left(\begin{array}{c}\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3t^2+\theta^3t+2\theta^2t+\theta^2+2\theta\right)e^{-\theta t}\right)\\+(\theta+1)\gamma(\beta+1,\theta t)\end{array}\right)}$$



The odds rate function of WMGL distribution for different values of θ and β.

For $\beta = 5$ and different values of θ, the hazard, cumulative hazard and odds funtions of the proposed distribution $(WMGL)$ are increasing functions. But, the reversed hazard is decreasing.

3.4 Order statistics and quantile function of WMGL distribution

This section is dedicated to deriving the distribution of order statistics and the quantile function associated with the weighted Mixed Gamma Lindley distribution.

3.4.1 Order statistics

Let $X_{(1)}, X_{(2)}..., X_{(n)}$ represent the order statistics of a random sample $X_{1,}X_2, ..., X_n$ drawn from the Weighted MGL distribution. The probability density function (pdf) of the i^{th} order statistic $X_{(i)}$ is derived by incorporating the cumulative distribution function (CDF) and pdf of the original distribution into the standard order statistics formula, which combines combinatorial terms and powers of the CDF and pdf.

$$g_{x_{(i)}}(x) = i\binom{n}{i}\left[\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)+(\theta+1)\gamma(\beta+1,\theta x)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right]^{i-1}$$

$$\times\left[1-\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)+(\theta+1)\gamma(\beta+1,\theta x)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right]^{n-i}$$

$$\times\frac{\theta^4\Gamma(\beta+1)(x+x^2)+\theta^{\beta+1}(\theta+1)x^\beta}{(\theta^2+3\theta+1)\Gamma(\beta+1)}e^{-\theta x}$$

$$= i\binom{n}{i}\left[\frac{1}{(\theta^2+3\theta+1)}\right]^{i-1}\left[\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)\right)+\left(\frac{(\theta+1)\gamma(\beta+1,\theta x)}{\Gamma(\beta+1)}\right)\right]^{i-1}$$

$$\times\left[\frac{1}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right]^{n-i}\left[((\theta+1)(\Gamma(\beta+1)-\gamma(\beta+1,\theta x)))+((\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta)e^{-\theta x})\right]^{n-i}$$

$$\times\frac{\theta^4\Gamma(\beta+1)(x+x^2)+\theta^{\beta+1}(\theta+1)x^\beta}{(\theta^2+3\theta+1)\Gamma(\beta+1)}e^{-\theta x}$$

$$= i\binom{n}{i}\left[\frac{1}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right]^{i-1}\left[\sum_{k=0}^{i-1}\binom{i-1}{k}\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)\right)^k\times\left(\frac{(\theta+1)\gamma(\beta+1,\theta x)}{\Gamma(\beta+1)}\right)^{i-1-k}\right]$$

$$\times\left[\frac{1}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right]^{n-i}\left[\sum_{k=0}^{n-i}\binom{n-i}{k}((\theta+1)(\Gamma(\beta+1)-\gamma(\beta+1,\theta x)))^k\times((\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta)e^{-\theta x})^{n-i-k}\right]$$

$$\times\frac{\theta^4\Gamma(\beta+1)(x+x^2)+\theta^{\beta+1}(\theta+1)x^\beta}{(\theta^2+3\theta+1)\Gamma(\beta+1)}e^{-\theta x}$$

3.4.2 Quantile function

The quantile function of the Weigthed Mixed Gamma Lindley distribution is calculeted by the following expression:

$$x = G^{-1}(x)$$

$$y = \frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right)+(\theta+1)\gamma(\beta+1,\theta x)}{\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)}$$

$$\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)\,y = \Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}\right) + (\theta+1)\gamma(\beta+1,\theta x)$$

$$\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)\,y-\Gamma(\beta+1)\left(\theta^2+2\theta\right) = (\theta+1)\gamma(\beta+1,\theta x) - \Gamma(\beta+1)\left(\theta^3x^2+\theta^3x+2\theta^2x+\theta^2+2\theta\right)e^{-\theta x}$$

3.5 Entropy measures

The Renyi and Tsallis entropy of the WMGL distribution model is defined in the followingsections:

3.5.1 Renyi entropy

The formula of the Renyi entropy is:

$$R_\lambda(X) = \frac{1}{1-\lambda}\ln\left[\int_0^\infty\left(\frac{\theta^4\Gamma(\beta+1)\left(x+x^2\right)+\theta^{\beta+1}(\theta+1)x^\beta}{\left(\theta^2+3\theta+1\right)\Gamma(\beta+1)}e^{-\theta x}\right)^\lambda\right]dx$$

$$= \frac{1}{1-\lambda}\ln\left[\begin{array}{c}\left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^\lambda\int_0^\infty\left(x+x^2\right)^\lambda e^{-\theta\lambda x}dx\\ +\left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^\lambda\int_0^\infty x^\beta e^{-\theta x}dx\end{array}\right]$$

$$= \frac{1}{1-\lambda}\ln\left[\begin{array}{c}\left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^\lambda\left(\int_0^\infty\sum_{i=1}^{\lambda}\binom{\lambda}{i}x^i x^{2(\lambda-i)}e^{-\theta\lambda x}dx\right)\\ +\left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^\lambda\int_0^\infty x^{\beta\lambda}e^{-\theta\lambda x}dx\end{array}\right]$$

$$= \frac{1}{1-\lambda}\ln\left[\begin{array}{c}\left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^\lambda\sum_{i=1}^{\lambda}\binom{\lambda}{i}\int_0^\infty x^{2\lambda-i}e^{-\theta\lambda x}dx\\ +\left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^\lambda\left(\frac{\Gamma(\beta\lambda+1)}{(\theta\lambda)^{(\beta\lambda+1)}}\right)\end{array}\right]$$

$$= \frac{1}{1-\lambda} \ln \left[\begin{array}{c} \left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^{\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \left(\frac{\Gamma(2\lambda-i+1)}{(\theta\lambda)^{(2\lambda-i+1)}}\right) \\ + \left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^{\lambda} \left(\frac{\Gamma(\beta\lambda+1)}{(\theta\lambda)^{(\beta\lambda+1)}}\right) \end{array} \right]$$

3.5.2 Tsallis entropy

The Tsallis entropy is expressed as:

$$T_{\lambda}(X) = \frac{1}{1-\lambda} \left[1 - \left(\int_0^{\infty} \left(\frac{\theta^4 \Gamma(\beta+1)(x+x^2) + \theta^{\beta+1}(\theta+1)x^{\beta}}{(\theta^2+3\theta+1)\Gamma(\beta+1)} e^{-\theta x} \right)^{\lambda} dx \right) \right]$$

$$= \frac{1}{1-\lambda} \left[1 - \left[\begin{array}{c} \left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^{\lambda} \int_0^{\infty} (x+x^2)^{\lambda} e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^{\lambda} \int_0^{\infty} x^{\beta} e^{-\theta x} dx \end{array} \right] \right]$$

$$= \frac{1}{1-\lambda} \left[1 - \left(\begin{array}{c} \left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^{\lambda} \left(\int_0^{\infty} \sum_{i=1}^{\lambda} \binom{\lambda}{i} x^i x^{2(\lambda-i)} e^{-\theta\lambda x} dx \right) \\ + \left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^{\lambda} \int_0^{\infty} x^{\beta\lambda} e^{-\theta\lambda x} dx \end{array} \right) \right]$$

$$= \frac{1}{1-\lambda} \left[1 - \left(\begin{array}{c} \left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^{\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \int_0^{\infty} x^{2\lambda-i} e^{-\theta\lambda x} dx \\ + \left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^{\lambda} \frac{\Gamma(\beta\lambda+1)}{(\theta\lambda)^{\beta\lambda+1}} \end{array} \right) \right]$$

$$= \frac{1}{1-\lambda} \left[1 - \left(\begin{array}{c} \left(\frac{\theta^4}{\theta^2+3\theta+1}\right)^{\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \left(\frac{\Gamma(2\lambda-i+1)}{(\theta\lambda)^{(2\lambda-i+1)}}\right) \\ + \left(\frac{\theta^{\beta+1}(\theta+1)}{\Gamma(\beta+1)}\right)^{\lambda} \frac{\Gamma(\beta\lambda+1)}{(\theta\lambda)^{\beta\lambda+1}} \end{array} \right) \right]$$

3.6 Methods of estimation

3.6.1 Maximum likelihood estimation

Assume that $X_{1,}X_{2,\cdots,}X_n$ is a random sample draw from WMGL distribution. The likelihood function $L(x;\beta,\theta)$, wich depends on the parameters θ *and* β,is given by:

$$L(x;\beta,\theta)=\prod_{i=1}^{n}\left[\frac{\theta^4\Gamma(\beta+1)(x+x^2)+\theta^{\beta+1}(\theta+1)x^\beta}{(\theta^2+3\theta+1)\Gamma(\beta+1)}e^{-\theta x}\right]$$

$$=\frac{e^{-\sum_{i=1}^{n}\theta x_i}}{(\theta^2+3\theta+1)^n(\Gamma(\beta+1))^n}\prod_{i=1}^{n}\left(\theta^4\Gamma(\beta+1)\left(x+x^2\right)+\theta^{\beta+1}(\theta+1)x^\beta\right)$$

$$\ln L(x;\beta,\theta)=-\sum_{i=1}^{n}\theta x_i-n\left[\ln\left(\theta^2+3\theta+1\right)+\ln(\Gamma(\beta+1))\right]$$
$$+\sum_{i=1}^{n}\ln\left(\theta^4\Gamma(\beta+1)\left(x_i+x_i^2\right)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)$$

$$\frac{\partial\ln(L(x;\beta,\theta))}{\partial\theta}=-\sum_{i=1}^{n}x_i-n\left[\frac{2\theta+3}{\theta^2+3\theta+1}\right]$$
$$+\sum_{i=1}^{n}\left[\frac{4\theta^3\Gamma(\beta+1)(x_i+x_i^2)+(\beta+1)\theta^\beta(\theta+1)x_i^\beta+\theta^{\beta+1}x_i^\beta}{\left(\theta^4\Gamma(\beta+1)(x_i+x_i^2)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\right]$$

The estimator of θ is calculated by solving

$$\frac{\partial\ln(L(x;\beta,\theta))}{\partial\theta}=0$$

$$-\sum_{i=1}^{n}x_i-n\left[\frac{2\theta+3}{\theta^2+3\theta+1}\right]+\sum_{i=1}^{n}\left[\frac{4\theta^3\Gamma(\beta+1)(x_i+x_i^2)+(\beta+1)\theta^\beta(\theta+1)x_i^\beta+\theta^{\beta+1}x_i^\beta}{\left(\theta^4\Gamma(\beta+1)(x_i+x_i^2)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\right]=0$$

$$\frac{\partial\ln(L(x;\beta,\theta))}{\partial\beta}=-n\left(\frac{\acute{\Gamma}(\beta+1)}{\Gamma(\beta+1)}\right)+\sum_{i=1}^{n}\left[\begin{array}{c}\frac{\theta^4\acute{\Gamma}(\beta+1)\left(x_i+x_i^2\right)}{\left(\theta^4\Gamma(\beta+1)\left(x_i+x_i^2\right)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\\+\frac{(\theta+1)x_i^\beta\ln\theta\times\theta^{\beta+1}+\theta^{\beta+1}(\theta+1)\ln x_i\times x_i^\beta}{\left(\theta^4\Gamma(\beta+1)\left(x_i+x_i^2\right)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\end{array}\right]$$

The β estimator' s is obtained by solving the correponding equation:

$$\frac{\partial\ln(L(x;\beta,\theta))}{\partial\beta}=0$$

$$-n\left(\frac{\acute{\Gamma}(\beta+1)}{\Gamma(\beta+1)}\right)+\sum_{i=1}^{n}\left[\begin{array}{c}\frac{\theta^4\acute{\Gamma}(\beta+1)\left(x_i+x_i^2\right)}{\left(\theta^4\Gamma(\beta+1)\left(x_i+x_i^2\right)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\\+\frac{(\theta+1)x_i^\beta\ln\theta\times\theta^{\beta+1}+\theta^{\beta+1}(\theta+1)\ln x_i\times x_i^\beta}{\left(\theta^4\Gamma(\beta+1)\left(x_i+x_i^2\right)+\theta^{\beta+1}(\theta+1)x_i^\beta\right)}\end{array}\right]=0$$

These equations are non linear, we need a simulation to find values of estimators.

3.6.2 Ordinary least squares estimation

The ordinary least squares estimator of θ and β is determined from the following expression:

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[G\left(x_{(i)}\right)-\frac{i}{n+1}\right]^{2}$$

The minimization of $T(x;\beta,\theta)$ function with respect to θ and β gives $\hat{\theta}$ and $\hat{\beta}$.

$$T(x;\beta,\theta)=\sum_{i=1}^{n}\left[\begin{array}{c}\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\\ -\frac{i}{n+1}\end{array}\right]^{2}$$

$$=\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)^{2}+\left(\frac{i}{n+1}\right)^{2}\\ -2\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)\times\left(\frac{i}{n+1}\right)\end{array}\right]$$

$$\frac{\partial T(x;\beta,\theta)}{\partial\beta}=\frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)^{2}\\ +\left(\frac{i}{n+1}\right)^{2}\\ -2\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)\\ \times\left(\frac{i}{n+1}\right)\end{array}\right]\right)$$

$$\frac{\partial T(x;\beta,\theta)}{\partial\theta}=\frac{\partial}{\partial\theta}\left(\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)^{2}\\ +\left(\frac{i}{n+1}\right)^{2}\\ -2\left(\frac{\Gamma(\beta+1)\left(\theta^{2}+2\theta-\left(\theta^{3}x_i^{2}+\theta^{3}x_i+2\theta^{2}x_i+\theta^{2}+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^{2}+3\theta+1)\Gamma(\beta+1)}\right)\\ \times\left(\frac{i}{n+1}\right)\end{array}\right]\right)$$

The explicit forms of, β and θ cannot be gained, so we may use some numerical methods to obtain them.

3.6.3 Weighted least squares estimations

The second method of least squares consists of minimizing the $W(x;\beta,\theta)$ function with respect to the parameters.

$$W(x;\beta,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

$$\times\left[\begin{array}{c}\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\\-\frac{i}{n+1}\end{array}\right]^2$$

$$=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

$$\times\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)^2\\+\left(\frac{i}{n+1}\right)^2\\-2\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)\\\times\left(\frac{i}{n+1}\right)\end{array}\right]$$

$$\frac{\partial W(x;\beta,\theta)}{\partial\beta}=$$

$$\frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)^2\\+\left(\frac{i}{n+1}\right)^2\\-2\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)\\\times\left(\frac{i}{n+1}\right)\end{array}\right]\right)$$

$$\frac{\partial W(x;\beta,\theta)}{\partial\theta}=$$

$$\frac{\partial}{\partial\theta}\left(\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\begin{array}{c}\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)^2\\+\left(\frac{i}{n+1}\right)^2\\-2\left(\frac{\Gamma(\beta+1)\left(\theta^2+2\theta-\left(\theta^3x_i^2+\theta^3x_i+2\theta^2x_i+\theta^2+2\theta\right)e^{-\theta x_i}\right)+(\theta+1)\gamma(\beta+1,\theta x_i)}{(\theta^2+3\theta+1)\Gamma(\beta+1)}\right)\\\times\left(\frac{i}{n+1}\right)\end{array}\right]\right)$$

Future works

This study offers a basis upon which further investagations can be developed. We can extand this work by

- Applying the quadratic transmutation and Top-leone method of the suggested models.

- Trying more real data sets.

Conclusion

In this dissertation, two continuous distributions are suggested using the mixture of distribution and weighted procedure. Some statistical features were obtained, the mean, standard deviation, skewness, kurtosis, and the coefficient of variation. We have adopted three methods of estimation to estimate the parameter space. Real data application were performed to show the effectiveness and potentiality between the first new distribution over many well-known distribution.

Chapter 2 presents Mixed Gamma Lindley distribution, as a mixture of Lindley (θ) and Gamma (β, θ) with mixing weights $\frac{\theta\beta}{\theta\beta+1}$ and $\frac{1}{\theta\beta+1}$, respectively. Some plots of the propability density function with different values of parameters and a table of some values of skewness and kurtosis are provided. They show that the MGL distribution is right skewed because of the positive values of skewness. Also, it is showed that the values of the mean and the standard deviation decrease as the values of θ increase and increase as values of β increases.

At the end of chapter, a real data application is applied to the proposed model to estimate the times failure data, then the results is compared with the results of some competitive distributions using some selection criteria (AIC, $CAIC$, BIC, $HQIC$, KS, $P-value$). The results of the analysis showed that MGL distribution is a best competitor and appropriate for fitting the given data than other models and this study is illustrated by two graphic illustrations.

In chapter 3, the weighted Mixed Gamma Lindley distribution was introduced as an extension of the MGL distribution. A pdf and cdf plots with some values of parameters β and θ, the mean, standard deviation and the coefficients of skewness, kurtosis and coefficient of variation are provided and giving some results which demonstrate that the probability density function is right skewed as the coefficient of skewness is positive. The distribution is platykurtic because the kurtosis is less than 3. The values of the mean and the standard deviation of the distribution increase as the values of the parameter β increase and decrease as parameter θ increase. The coefficients of skeweness, kurtosis and variation decrease as the values of parameters β and θ

incraese.

Finally, this chapter is ended by some future works which can be a starting poit for further studies.

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Appendix A: Abbreviations and Notations

The different abbreviations and notations used throughout this dissertation are explained below:

μ	:	Mean of population
σ_2	:	Variance of population
σ	:	Sandard deviation
$\mathrm{M}_X(t)$	:	Moment generating function.
L	:	Likelihood function.
Θ	:	Parameter space.
$\mathrm{F}_n(X)$	:	Empirical distribution function.
PDF	:	Probability density function.
CDF	:	Cumulative distribution function.
MGL	:	Mixed Gamma Lindley
WMGL	:	Weighted Mixed Gamma Lindley
Γ	:	Gamma function
Sk	:	Skewness
Kur	:	Kurtosis
CV	:	Coefficient of variation.

MLE	:	Maximum likelihood estimater.
OLS	:	Ordinary least square estimator.
WLS	:	weighted least square estimator.
AIC	:	Akaike information criterion
CAIC	:	coreected Akaike information criterion
BIC	:	Bayesian information criterion
HQIC	:	Hannan-Quinn information criterion
KS	:	Kolmogorov Smirnov.
Gamma(.,.)	:	Gamma distribution.