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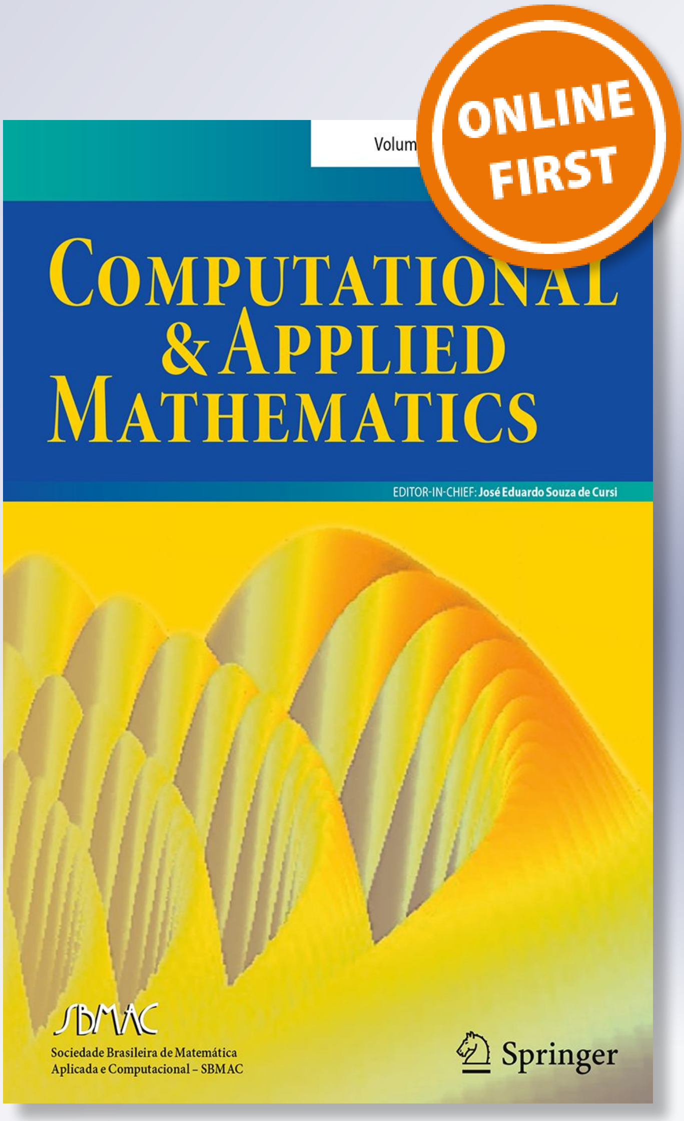

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On the regularization method for Fredholm integral equations with odd weakly singular kernel

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Abstract In this paper, we propose a numerical method to approach the solution of a Fredholm integral equation with a weakly singular kernel by applying the convolution product as a regularization operator and the Fourier series as a projection. Preliminary numerical results show that the order of convergence of the method is better than the one of conventional projection methods.

Keywords Integral equations · Weak singularity · Convolution · Fourier series

Mathematics Subject Classification 45B05 · 45E99 · 65R20 · 65T40

1 Introduction

The theory of integral equations has been an active search domain for many years and is based on analysis, function theory and functional analysis. On the other hand, integral equations are of practical interest because of the approximation theory of Fredholm integral equations, which allows the application of the notions of operator theory, spectral theory and especially those of projection and functional approximation (Debbar et al. 2016; Guebbai and Grammont 2014; Ahues et al. 2009; Amosov and Youssef 2016; Lemita and Guebbai 2017).

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The approximation of Fredholm weakly singular equations of the following form: For all $f \in L^1(0,1)$, find $u \in L^1(0,1)$, such that

$$\forall s \in [0,1], \ u(s) = \int_0^1 g(|s-t|)\, u(t) \mathrm{d}t + f(s),$$

where, g, the source of the weak singularity, is supposed to be in $L^1([0,1],\mathbb{R})$, positive, decreasing and such that

$$\lim_{t \to 0^+} g(t) = +\infty,$$

knows an impassable error given by: $\exists C_1, C_2 > 0$, for $n \in \mathbb{N}$ large enough,

$$\|u - u_n\|_{L^1(0,1)} \leq C_1 \int_0^{\frac{1}{n}} g(t)\, \mathrm{d}t \leq C_2 \left(\frac{1}{n}\right)^{\gamma},$$

where γ depends on g and belongs to $(0,1)$ for some interesting cases like Abel's kernel.

Recently, Debbar et al, have improved this bound, by adding rather acceptable conditions on the solution u, to obtain a convergence order strictly greater than γ and at most equal to 1.

In this paper, we are interested in the same integral equation in the case of an odd kernel i.e. g verifies

$$\begin{array}{ll} (1) & g \in L^1([-1,1],\mathbb{R}), \\ (2) & g(-t) = -g(t), \ t \in [-1,1], \\ (3) & g(t) \geq 0, \ t \in [0,1], \\ (4) & g \text{ is decreasing over } [0,1], \\ (5) & \lim_{t \to 0^+} g(t) = +\infty. \end{array}$$

This version was studied numerically in Guebbai and Grammont (2014) and the authors obtained a convergence order equal to γ. Our goal is to show that the method developed in Debbar et al. (2016) is applicable in this case and that we can improve the order of convergence obtained before.

2 Setting the problem and studying its regularity

Let $\mathcal{B}$ be the Banach space of all bounded linear operators defined from $L^1([0,1],\mathbb{C})$ into itself. The norm of $\mathcal{B}$ is given by :

$$\forall A \in \mathcal{B}, \quad \|A\|_{\mathcal{B}} = \sup_{\|x\|_{L^1([0,1],\mathbb{C})}=1} \|Ax\|_{L^1([0,1],\mathbb{C})}.$$

Let $g \in L^1([-1,1],\mathbb{R})$ be an odd function, positive and decreasing over $(0,1)$ such that:

$$\lim_{t \to 0^+} g(t) = +\infty.$$

Let K be a linear operator defined from $L^1([0,1],\mathbb{C})$ into itself by

$$\forall x \in L^1([0,1],\mathbb{C}), \quad Kx(s) := \int_0^1 g(s-t)x(t)\mathrm{d}t.$$

It is easy to establish that $K \in \mathcal{B}$ and $\|K\|_{\mathcal{B}} \leq \|g\|_{L^1([-1,1],\mathbb{R})}$. The operator K is an integral operator with a weakly singular kernel. The singularity comes from the fact that

$$\lim_{t \to 0^+} g(t) = +\infty.$$

But it remains weak because

$$\int_{-1}^{1} |g\,(t)|\,\mathrm{d}t < +\infty.$$

Let I denotes the identity operator in $L^1\,([0,1]\,,\mathbb{C})$. We suppose that $(I-K)^{-1}$ exists and that

$$d := \left\|(I-K)^{-1}\right\|_{\mathcal{B}} < \infty.$$

Then, for all $f \in L^1\,([0,1]\,,\mathbb{C})$, the equation

$$u = Ku + f,$$

has a unique solution $u \in L^1\,([0,1]\,,\mathbb{C})$. Before describing our approach let us state a regularity result.

Theorem 1 *Let $x \in L^1\,([0,1]\,,\mathbb{C})$ be such that $x' \in L^1\,([0,1]\,,\mathbb{C})$. Then Kx is a differentiable function at all $s \in (0,1)$, and*

$$(Kx)'(s) = x(0)g(s) + x(1)g(1-s) + (Kx')(s),\ \ s \in (0,1)\,.$$

In addition,

$$\left\|(Kx)'\right\|_{L^1([0,1],\mathbb{C})} \leq 2\|g\|_{L^1([-1,1],\mathbb{C})}\left(\|x\|_{L^1([0,1],\mathbb{C})} + \left\|x'\right\|_{L^1([0,1],\mathbb{C})}\right).$$

Proof We use an idea developed in Ahues et al. (2009). For $0 < \eta < \frac{1}{2}$, we define the linear operator $K_\eta : L^1\,([0,1]\,,\mathbb{C}) \to L^1\,([0,1]\,,\mathbb{C})$ by: For $x \in L^1\,([0,1]\,,\mathbb{C})$,

$$K_\eta x\,(s) := \begin{cases} \displaystyle\int_0^{s-\eta} g(s-t)x(t)\mathrm{d}t - \int_{s+\eta}^{1} g(t-s)x(t)\mathrm{d}t, & s \in (\eta, 1-\eta)\,, \\ \displaystyle\int_0^{s-\eta} g(s-t)x(t)\mathrm{d}t, & s \in [1-\eta, 1)\,, \\ \displaystyle-\int_{s+\eta}^{1} g(t-s)x(t)\mathrm{d}t, & s \in (0,\eta]\,. \end{cases}$$

Changing of variables, we get

$$K_\eta x\,(s) := \begin{cases} \displaystyle\int_\eta^{s} g(u)x(s-u)\mathrm{d}u - \int_{\eta}^{1-s} g(v)x(s+v)\mathrm{d}t, & s \in (\eta, 1-\eta)\,, \\ \displaystyle\int_\eta^{s} g(u)x(s-u)\mathrm{d}u, & s \in [1-\eta, 1)\,, \\ \displaystyle-\int_{\eta}^{1-s} g(v)x(s+v)\mathrm{d}t, & s \in (0,\eta]\,. \end{cases}$$

Suppose that $x' \in L^1([0,1],\mathbb{C})$. Then,

$$\left(K_\eta x\right)'(s) := \begin{cases} x(0)g(s) + \int_\eta^s g(u)x'(s-u)\mathrm{d}u + x(1)g(1-s) & \\ -\int_\eta^{1-s} g(v)x'(s+v)\mathrm{d}t, & s \in (\eta, 1-\eta), \\ x(0)g(s) + \int_\eta^s g(u)x'(s-u)\mathrm{d}u, & s \in [1-\eta, 1), \\ x(1)g(1-s) - \int_\eta^{1-s} g(v)x'(s+v)\mathrm{d}t, & s \in (0, \eta]. \end{cases}$$

It can easily be checked that

$$\lim_{\eta\to 0} \left\| Kx - K_\eta x \right\|_{L^1([0,1],\mathbb{C})} = 0,$$

$$\forall s \in (0,1), \lim_{\eta\to 0} \left| \left(K_\eta x\right)'(s) - \left(x(0)g(s) + x(1)g(1-s) + (Kx')(s)\right) \right| = 0.$$

This proves that $(Kx)'(s) = x(0)g(s) + x(1)g(1-s) + (Kx')(s),\ s \in (0,1)$. We have

$$x(0) = \int_0^1 ((1-t)\,x(t))'\,\mathrm{d}t \Rightarrow |x(0)| \le \|x\|_{L^1([0,1],\mathbb{C})} + \left\|x'\right\|_{L^1([0,1],\mathbb{C})},$$

$$x(1) = \int_0^1 (tx(t))'\,\mathrm{d}t \Rightarrow |x(1)| \le \|x\|_{L^1([0,1],\mathbb{C})} + \left\|x'\right\|_{L^1([0,1],\mathbb{C})}.$$

Then

$$\left\|(Kx)'\right\|_{L^1([0,1],\mathbb{C})} \le 2\,\|g\|_{L^1([-1,1],\mathbb{R})} \left(\|x\|_{L^1([0,1],\mathbb{C})} + \left\|x'\right\|_{L^1([0,1],\mathbb{C})} \right).$$

□

3 Regularization

In this article, we propose a new approach that differs from Guebbai and Grammont (2014) which relied on the convolution product as follows : $\forall x \in L^1([0,1],\mathbb{C})$, Kx is extended as

$$Kx(s) := -Kx(-s),\ s \in [-1,0].$$

For $\varepsilon > 0$, we define K_ε by

$$\forall x \in L^1([0,1],\mathbb{C}),\quad K_\varepsilon x := J_\varepsilon * Kx,$$

where J is a function defined on $\mathbb{R}$ satisfying the following properties:

$$\begin{cases} J \text{ is even}, \\ J(x) \ge 0\ \forall x \in \mathbb{R}, \\ J \in C^m(\mathbb{R},\mathbb{R}),\quad m > 1, \\ \operatorname{Supp}(J) \subset [-1,1], \\ \int_{-1}^1 J(\tau)\,\mathrm{d}\tau = 1, \end{cases}$$

and for $\varepsilon > 0$,

$$J_\varepsilon(\tau) := \varepsilon^{-1} J\left(\varepsilon^{-1}\tau\right).$$

Hence, for all $x \in L^1([0,1],\mathbb{C})$, $Kx \in L^1([0,1],\mathbb{C})$, and $K_\varepsilon x \in L^1([0,1],\mathbb{C})$.

For all $t \in [0,1]$, we define $g_t(s) := g(s-t)$. g_t is extended as follows $g_t(s) := -g_t(-s)$ for all $s \in [-1,0]$. Then g_t is 2-periodical on $\mathbb{R}$.

$\forall \varepsilon > 0$, we define:

$$g_\varepsilon(s,t) = J_\varepsilon * g_t(s).$$

We can see that $g_\varepsilon(.,t) \in C^m_{2per}(\mathbb{R},\mathbb{R})$, Adams (1975). This extension is similar to that of Debbar et al. (2016).

Proposition 1 *K_ε is an integral operator. For all $x \in L^1([0,1],\mathbb{C})$,*

$$K_\varepsilon x(s) = \int_0^1 g_\varepsilon(s,t)x(t)dt.$$

Proof For all $x \in L^1([0,1],\mathbb{C})$,

$$\begin{aligned}
K_\varepsilon x(s) &= J_\varepsilon * Kx(s),\\
&= \int_{-\varepsilon}^{\varepsilon} J_\varepsilon(y)\, Kx(s-y)\mathrm{d}y,\\
&= \int_{-1}^{1} \varepsilon J_\varepsilon(\varepsilon\tau)\, Kx(s-\varepsilon\tau)\,\mathrm{d}\tau,\\
&= \int_{-1}^{1} J(\tau) Kx(s-\varepsilon\tau)\mathrm{d}\tau,\\
&= \int_{-1}^{1} J(\tau) \int_0^1 g(s-\varepsilon\tau - t)\, x(t)\mathrm{d}t\mathrm{d}\tau,\\
&= \int_0^1 \left[\int_{-1}^{1} J(\tau) g(s-\varepsilon\tau-t)\,\mathrm{d}\tau\right] x(t)\mathrm{d}t,\\
&= \int_0^1 g_\varepsilon(s,t)x(t)\mathrm{d}t.
\end{aligned}$$

□

Proposition 2

$$\lim_{\varepsilon\to 0^+} \|K - K_\varepsilon\|_{\mathcal{B}} = 0.$$

Proof Let $u \in L^1([0,1],\mathbb{C})$ be such that $\|u\|_{L^1([0,1],\mathbb{C})} = 1$. Then

$$\begin{aligned}
\|Ku - K_\varepsilon u\|_{L^1([0,1],\mathbb{C})} &= \int_0^1 |K_\varepsilon u(s) - Ku(s)|\,\mathrm{d}s,\\
&= \int_0^1 \left|\int_0^1 g_\varepsilon(s,t)u(t)\mathrm{d}t - \int_0^1 g(|s-t|)u(t)\mathrm{d}t\right| \mathrm{d}s,\\
&\le \int_0^1 \int_0^1 |J_\varepsilon * g_t(s) - g_t(s)|\,|u(t)|\mathrm{d}t\mathrm{d}s,\\
&\le \left[\sup_{t\in[0,1]} \int_0^1 |J_\varepsilon * g_t(s) - g_t(s)|\mathrm{d}s\right] \underbrace{\int_0^1 |u(t)|\mathrm{d}t}_{=1},
\end{aligned}$$

$$\leq \sup_{t\in[0,1]} \|J_\varepsilon * g_t - g_t\|_{L^1([0,1])} \overset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

□

We will study the invertibility of $I - K_\varepsilon$.

Lemma 1 *For $\varepsilon > 0$ small enough, $(I - K_\varepsilon)^{-1}$ exists and*

$$\|(I - K_\varepsilon)^{-1}\|_{\mathcal{B}} \leq 2d.$$

Proof We have,

$$I - K_\varepsilon = (I - K)(I - (I - K)^{-1}(K_\varepsilon - K)),$$

and,

$$\lim_{\varepsilon\to 0^+} \|(I - K)^{-1}(K_\varepsilon - K)\|_{\mathcal{B}} = 0.$$

Where, for ε sufficiently small,

$$\|(I - K)^{-1}(K_\varepsilon - K)\|_{\mathcal{B}} < 1,$$

and $I - (I - K)^{-1}(K_\varepsilon - K)$ is invertible. By applying Neumann's theorem, we find that $(I - K_\varepsilon)^{-1}$ exists. Moreover,

$$\|(I - (I - K)^{-1}(K_\varepsilon - K))^{-1}\|_{\mathcal{B}} \leq \frac{1}{1 - \|(I - K)^{-1}(K_\varepsilon - K)\|_{\mathcal{B}}} \overset{\varepsilon\to 0^+}{\longrightarrow} 1.$$

Then for ε sufficiently small, we have:

$$\|(I - (I - K)^{-1}(K_\varepsilon - K))^{-1}\|_{\mathcal{B}} \leq 2,$$

and,

$$\|(I - K_\varepsilon)^{-1}\|_{\mathcal{B}} \leq 2d.$$

□

Knowing that $I - K_\varepsilon$ is invertible, for $\varepsilon > 0$ small enough, we note u_ε the unique function in $L^1([0, 1], \mathbb{C})$ which satisfies $u_\varepsilon = K_\varepsilon u_\varepsilon + f$.

Theorem 2 *Suppose that $u' \in L^1([0, 1], \mathbb{C})$, then for ε small enough,*

$$\|u - u_\varepsilon\|_{L^1([0,1],\mathbb{C})} \leq 2d \left(2 \|g\|_{L^1([-1,1],\mathbb{R})} \left(\|u\|_{L^1([0,1],\mathbb{C})} + \left\|u'\right\|_{L^1([0,1],\mathbb{C})}\right) + 1\right) \varepsilon.$$

Proof We have,

$$\begin{aligned} u - u_\varepsilon &= Ku - K_\varepsilon u_\varepsilon \\ &= (I - K_\varepsilon)^{-1}(Ku - K_\varepsilon u) \\ &= (I - K_\varepsilon)^{-1}(Ku - J_\varepsilon * Ku). \end{aligned}$$

But

$$\begin{aligned}\|J_\varepsilon * Ku - Ku\|_{L^1([0,1],\mathbb{R})} &= \int_0^1 \left| \int_{-1}^1 J(\tau) Ku(s-\varepsilon\tau)\mathrm{d}\tau - \int_{-1}^1 J(\tau) Ku(s)\mathrm{d}\tau \right| \mathrm{d}s,\\ &\leq \max_{\tau\in[-1,1]} J(\tau) \int_0^1 \int_{-1}^1 |Ku(s-\varepsilon\tau) - Ku(s)|\,\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

According to Theorem 1 and the intermediate value theorem for derivatives

$$\exists \theta \in]-\varepsilon, 0[, \quad Ku(s-\varepsilon\tau) - Ku(s) = (Ku)'(s+\theta\tau)(-\varepsilon\tau).$$

We have, $\forall s \in]0,1[, \forall \tau \in]-1,1[$,

$$\lim_{\varepsilon\to 0} (Ku)'(s+\theta\tau) = (Ku)'(s).$$

Then, for ε small enough, $\left|(Ku)'(s+\theta\tau)\right| \leq \left|(Ku)'(s)\right| + 1$. Using Lebesgue's theorem, we obtain

$$\lim_{\varepsilon\to 0^+} \int_0^1 \int_{-1}^1 \left|(Ku)'(s+\theta\tau)\right| |\tau|\,\mathrm{d}\tau\mathrm{d}s = \left\|(Ku)'\right\|_{L^1([0,1],\mathbb{C})}.$$

Then, for ε small enough, we obtain

$$\begin{aligned}\|Ku - J_\varepsilon * Ku\|_{L^1([0,1],\mathbb{C})} &\leq \left(\left\|(Ku)'\right\|_{L^1([0,1],\mathbb{C})} + 1\right)\varepsilon\\ &\leq \left(2\|g\|_{L^1([-1,1],\mathbb{C})}\left(\|u\|_{L^1([0,1],\mathbb{C})} + \left\|u'\right\|_{L^1([0,1],\mathbb{C})}\right) + 1\right)\varepsilon.\end{aligned}$$

We use Lemma 1 to get the result. □

In what follows, the solution is supposed to verify the following conditions:

$$u \in L^1([0,1],\mathbb{C}) \text{ and } u' \in L^1([0,1],\mathbb{C}).$$

4 Degenerate kernel

In this section, we rely on the kernel of operator K_ε to propose a method to build an easy-to-use numerical approximation of the solution u_ε.

4.1 Projection

We note that, for all $\varepsilon > 0$, and for all $x \in L^1([0,1],\mathbb{R})$, $K_\varepsilon x(s)$ is odd and 2-periodical. Thus, for all $N \in \mathbb{N}^*$, we define the operators $K_{\varepsilon,N}$ from $L^1([0,1],\mathbb{C})$ into itself by:

$$\forall x \in L^1([0,1],\mathbb{R}),\ K_{\varepsilon,N}x := P_N K_\varepsilon x := \sum_{k=1}^N b_{k,\varepsilon}\sin(k\pi\cdot),$$

where,

$$b_{k,\varepsilon} := \int_0^2 K_\varepsilon x(s)\sin(k\pi s)\mathrm{d}s.$$

P_N represents the Fourier series expansion truncated to the order N , Zygmund (1959)

Proposition 3 *For all $\varepsilon > 0$, and for all $N \geq 2$, $K_{\varepsilon,N}$ is an integral operator with a degenerate kernel.*

Proof For all $x \in L^1([0,1],\mathbb{C})$,

$$K_{\varepsilon,N}x(s) = \sum_{k=1}^{N} b_{k,\varepsilon}\sin(k\pi s),$$

where

$$\begin{aligned} b_{k,\varepsilon} &= \int_0^2 K_\varepsilon x(s)\sin(k\pi s)\mathrm{d}s \\ &= \int_0^2 \left[\int_0^1 g_\varepsilon(s,t)x(t)\mathrm{d}t\right]\sin(k\pi s)\mathrm{d}s \\ &= \int_0^1 \left[\int_0^2 g_\varepsilon(s,t)\sin(k\pi s)\mathrm{d}s\right]x(t)\mathrm{d}t. \end{aligned}$$

We set

$$\varphi_{\varepsilon,k}(t) = \int_0^2 g_\varepsilon(s,t)\sin(k\pi s)\mathrm{d}s.$$

Thus,

$$K_{\varepsilon,N}x(s) = \sum_{k=1}^{N} \psi_k(s)\int_0^1 \varphi_{\varepsilon,k}(t)\,x(t)\mathrm{d}t.$$

Where

$$\psi_k(s) = \sin(k\pi s),\ 1 \le k \le N.$$

□

Theorem 3 *For all* $\varepsilon > 0$, *and for all* $N > 2$,

$$\left\|K_\varepsilon - K_{\varepsilon,N}\right\|_{\mathcal{B}} \le \kappa_m \left\|J^{(m)}\right\|_{L^1([0,1],\mathbb{R})} \|g\|_{L^1([-1,1],\mathbb{R})}\,\varepsilon^{-m}(3+\ln N)N^{-m},$$

where κ_m *is the Favard constant.*

Proof We have

$$\left\|K_\varepsilon - K_{\varepsilon,N}\right\|_{\mathcal{B}} \le \sup_{t\in[0,1]} \left\|g_\varepsilon(.,t) - g_{\varepsilon,N}(.,t)\right\|_{L^1([-1,1],\mathbb{R})}.$$

Let τ_N be the subspace of $L^1([0,1],\mathbb{R})$ generated by the family $\{\sin(k\pi.)\}_{1\le k\le N}$. We have

$$g_{\varepsilon,N}(.,t) = P_N g_\varepsilon(.,t)$$

For all $t \in [0,1]$,

$$g_\varepsilon(.,t) - g_{\varepsilon,N}(.,t) = (I - P_N)\left(g_\varepsilon(.,t) - \upsilon_N(.)\right) \text{ for all } \upsilon_N \in \tau_N.$$

As

$$\begin{aligned} \left\|g_\varepsilon(.,t) - g_{\varepsilon,N}(.,t)\right\|_{L^1([-1,1],\mathbb{R})} &= \|g_\varepsilon(.,t) - P_N g_\varepsilon\|_{L^1([-1,1],\mathbb{R})} \\ &= \|(I - P_N)\,g_\varepsilon(.,t)\|_{L^1([-1,1],\mathbb{R})} \\ &= \|(I - P_N)\,(g_\varepsilon(.,t) - \upsilon_N(.))\|_{L^1([-1,1],\mathbb{R})} \\ &\le (1 + \|P_N\|_{\mathcal{B}}) \inf_{\upsilon_N\in\tau_N} \|g_\varepsilon(.,t) - \upsilon_N\|_{L^1([-1,1],\mathbb{R})}, \end{aligned}$$

where $\upsilon_N \in \tau_N$.

Here, $\|P_N\|_{\mathcal{B}} \leq (2 + \ln N)$, Davidson and Donsig (2009). Applying Nikolskii's theorem Tikhomirov (1976), we get

$$\inf_{\upsilon_N \in \tau_N} \|g_\varepsilon(.,t) - \upsilon_N\|_{L^1([-1,1],\mathbb{R})} \leq \kappa_m \pi^{-m} N^{-m} \left\| \frac{\partial^m g_\varepsilon}{\partial s^m}(.,t) \right\|_{L^1([-1,1],\mathbb{R})},$$

where

$$\kappa_m = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}}.$$

But,

$$\begin{aligned} \left\| \frac{\partial^m g_\varepsilon}{\partial s^m}(.,t) \right\|_{L^1([-1,1],\mathbb{R})} &= \int_0^1 \left| \int_{-\varepsilon}^{\varepsilon} J_\varepsilon^{(m)}(\tau)\, g(s - \tau - t)\, \mathrm{d}\tau \right| \mathrm{d}s \\ &\leq \int_0^1 \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon^{m+1}} J^{(m)}\left(\frac{\tau}{\varepsilon}\right) |g(s - \tau - t)|\, \mathrm{d}\tau \mathrm{d}s \\ &\leq \frac{1}{\varepsilon^m} \int_0^1 \int_{-1}^{1} J^{(m)}(\tau)\, |g(s - \varepsilon\tau - t)|\, \mathrm{d}\tau \mathrm{d}s \\ &\leq \varepsilon^{-m} \left\| J^{(m)} \right\|_{L^1([-1,1],\mathbb{R})} \|g\|_{L^1([-1,1],\mathbb{C})}. \end{aligned}$$

□

Lemma 2 *For N large enough, $\left(I - K_{\varepsilon,N}\right)^{-1}$ exists and*

$$\left\|(I - K_{\varepsilon,N})^{-1}\right\|_{\mathcal{B}} \leq 4d.$$

Proof Reasoning follows as in Lemma 1. □

Because $\left(I - K_{\varepsilon,N}\right)^{-1}$is invertible for N large enough then the equation

$$u_{\varepsilon,N} = K_{\varepsilon,N} u_{\varepsilon,N} + f$$

has a unique solution in $L^1([0,1],\mathbb{C})$.

Theorem 4 *There exist $\theta > 0$, $\beta > 0$, $C > 0$ such that, for $\varepsilon > 0$ and N large enough,*

$$\frac{\left\|u_\varepsilon - u_{\varepsilon,N}\right\|_{L^1([0,1],\mathbb{C})}}{\|u_\varepsilon\|_{L^1([0,1],\mathbb{C})}} \leq C\varepsilon^{-m} (\theta + \beta \ln N)\, N^{-m}.$$

Proof The proof is as in Theorem 2. □

4.2 Approximate solution

We define

$$\alpha := \frac{m}{m+1}, \quad \varepsilon := N^{-\alpha}, \quad K_N := K_{\varepsilon,N} = K_{N^{-\alpha},N}.$$

We approximate the solution u by solving equation

$$u_N - K_N u_N = f.$$

Proposition 4

$$\lim_{N\to\infty} \|K - K_N\|_{\mathcal{B}} = 0.$$

Proof We have

$$\|K - K_N\|_{\mathcal{B}} \le \|K - K_{N^{-\alpha}}\|_{\mathcal{B}} + \|K_{N^{-\alpha}} - K_N\|_{\mathcal{B}}.$$

Using Theorem 3, we get

$$\|K_{N^{-\alpha}} - K_N\|_{\mathcal{B}} \le 2\kappa_m \left\|J^{(m)}\right\|_{L^1([0,1],\mathbb{R})} \|g\|_{L^1([0,1],\mathbb{R})} (3 + \ln N) N^{-\frac{m}{m+1}}.$$

Using Proposition 2,

$$\lim_{N\to\infty} \|K - K_{N^{-\alpha}}\|_{\mathcal{B}} = 0.$$

□

Lemma 3 *For N large enough, $(I - K_N)^{-1}$ exists and*

$$\left\|(I - K_N)^{-1}\right\|_{\mathcal{B}} \le 2d.$$

Proof As in Lemma 1. □

Let u_N be the unique solution of the equation

$$u_N - K_N u_N = f.$$

Theorem 5 *There exist $\theta_1 > 0,\ \beta_1 > 0$ such that, for N large enough,*

$$\|u - u_N\|_{L^1([0,1],\mathbb{C})} \le (\theta_1 + \beta_1 \ln(N))\, N^{-\frac{m}{m+1}}.$$

Proof We have,

$$\|u - u_N\|_{L^1([0,1],\mathbb{C})} \le \|u - u_{N^{-\alpha}}\|_{L^1([0,1],\mathbb{C})} + \|u_{N^{-\alpha}} - u_N\|_{L^1([0,1],\mathbb{C})}.$$

We use Theorems 2 and 4 to conclude. □

$\theta_1,\ \beta_1$ depend on $u,\ u'$ and $u_{N^{-\alpha}}$. The next result is very close to the concept of the relative error in our case:

Proposition 5 *There exist $\theta_2 > 0,\ \beta_2 > 0$ such that, for N large enough,*

$$\frac{\|u - u_N\|_{L^1([0,1],\mathbb{C})}}{\max\left(1, \|u\|_{L^1([0,1],\mathbb{C})} + \|u'\|_{L^1([0,1],\mathbb{C})}\right)} \le (\theta_2 + \beta_2 \ln(N))\, N^{-\frac{m}{m+1}}.$$

Proof It suffices to see that for N large enough,

$$\|u_{N^{-\alpha}}\|_{L^1([0,1],\mathbb{C})} \le 1 + \|u\|_{L^1([0,1],\mathbb{C})}.$$

We conclude in the same way as in the last theorem. □

Table 1 Relative error ($m = 2$)

N	M_N	$\frac{\|u-u_N\|_{L^1([0,1],\mathbb{C})}}{\|u\|_{L^1([0,1],\mathbb{C})}}$
50	50	9.4E−1
100	80	4.2E−1
200	180	1.5E−1
250	240	6.2E−2
300	280	3.9E−2

Table 2 Relative error ($m = 4$)

N	M_N	$\frac{\|u-u_N\|_{L^1([0,1],\mathbb{C})}}{\|u\|_{L^1([0,1],\mathbb{C})}}$
50	50	7.4E−2
100	80	3.8E−2
200	180	1.4E−2
250	240	8.4E−3
300	280	3.6E−3

5 Numerical results

Using Mathematica from [11], alculate a primitive of $\tau \to J(\tau)g_t(s-\varepsilon\tau)$. The result is introduced in the solver, which will choose at each t and s, the proper integral limits to calculate $g_\varepsilon(s,t)$. Since $u_N = K_N u_N + f$ for all $s \in [0,1]$,

$$u_N(s) = \sum_{j=1}^{N} \psi_j(s) \int_0^1 \varphi_{N^{-\alpha},j}(t)\, u_N(t)\mathrm{d}t + f(s).$$

Multiplying by $\varphi_{N^{-\alpha},j}(\cdot)$ and integrating over $[0,1]$, we get

$$\mathsf{x} = \mathsf{A}\mathsf{x} + \mathsf{b},$$

where,

$$\mathsf{x}(i) := \int_0^1 \varphi_{N^{-\alpha},i}(t)\, u_N(t)\mathrm{d}t, \quad \mathsf{b}(i) := \int_0^1 \varphi_{N^{-\alpha},i}(t)\, f(t)\mathrm{d}t, \quad 1 \le i \le N,$$
$$\mathsf{A}(i,j) := \int_0^1 \varphi_{N^{-\alpha},i}(t)\, \psi_j(t)\mathrm{d}t, \quad 1 \le i \le N, \quad 1 \le j \le N.$$

The integrals involved in the computation of A and b, are computed numerically using the composite trapezoidal rule with a number M_N of nodes depending on N (see Tables 1, 2). M_N is chosen in such a way that the error introduced by the composite trapezoidal rule does not affect the global order of convergence of the Fourier series approach. Once the linear system is solved, u_N is built as

$$u_N = \sum_{j=0}^{N} \mathsf{x}(j)\psi_j + f.$$

To test the efficiency of this method, we choose the integral equation with the following Abel's kernel:

$$g(t) := \frac{1}{3}t^{-\frac{1}{5}}, \qquad t \in \,]0, 1],$$
$$g(t) := -\frac{1}{3}t^{-\frac{1}{5}}, \qquad t \in [-1, 0[,$$

and,

$$f(s) = \begin{cases} \frac{5}{12}(1-s)^{\frac{4}{5}} - \frac{5}{12}\left(\frac{1}{2}-s\right)^{\frac{4}{5}} & s \in \left[0, \frac{1}{2}\right], \\ 1 + \frac{5}{12}(1-s)^{\frac{4}{5}} - \frac{5}{12}\left(s-\frac{1}{2}\right)^{\frac{4}{5}} & s \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

Then, the exact solution is

$$u(s) := \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right], \\ 1 & s \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

We remark that $u' = \delta_{\frac{1}{2}} \in L^1([0,1], \mathbb{C})$, where δ is the Dirac's distribution, $\|u\|_{L^1([0,1],\mathbb{C})} = 1/2$ and

$$\max\left(1, \|u\|_{L^1([0,1],\mathbb{C})} + \left\|u'\right\|_{L^1([0,1],\mathbb{C})}\right) = \frac{3}{2}.$$

The relative error, $\frac{\|u_n - u\|_{L^1([0,1],\mathbb{C})}}{\|u\|_{L^1([0,1],\mathbb{C})}} = 2\|u_n - u\|_{L^1([0,1],\mathbb{C})}$ is computed numerically using a trapezoidal composite rule with 1000 nodes.

First, we use the following regularizing function,

$$J(\tau) := \begin{cases} \frac{35}{32}(1-\tau^2)^3 & \text{for } \tau \in [-1, 1], \\ 0 & \text{for } \tau \notin [-1, 1]. \end{cases}$$

Here, $m = 2$.

Second, we use the following regularizing function,

$$J(\tau) := \begin{cases} \frac{693}{512}(1-\tau^2)^5 & \text{for } \tau \in [-1, 1], \\ 0 & \text{for } \tau \notin [-1, 1]. \end{cases}$$

Here, $m = 4$.

6 Conclusions

We have built a numerical approximation method for Fredholm integral equations in two steps: We use the convulotion product as a regularization operator to get a new integral operator with a smooth kernel. Then we build a degenerate kernel operator using the Fourier series expansion.

Under the condition

$$u \in L^1([0,1], \mathbb{C}) \quad \text{and} \quad u' \in L^1([0,1], \mathbb{C}),$$

this method provides a better convergence order. Indeed, methods constructed in Guebbai and Grammont (2014) give

$$\frac{\|u-u_N\|_{\mathrm{L}^1([0,1],\mathbb{C})}}{\|u\|_{\mathrm{L}^1([0,1],\mathbb{C})}} \leq \beta N^{-\frac{\gamma(m-1)}{m+\gamma}}$$

under the following condition: $\exists C>0, \exists \gamma \in (0,1)$ such that, for $h>0$ small enough,

$$\int_0^h g(t)\mathrm{d}t \leq Ch^{\gamma}.$$

It is clear that,

$$\frac{\gamma(m-1)}{m+\gamma} < \frac{m}{m+1}.$$

In addition, classical projection methods described in Ahues et al. (2009) and Atkinson and Han (2001) give

$$\frac{\|u-u_h\|_{L^1([0,1],\mathbb{C})}}{\|u\|_{L^1([0,1],\mathbb{C})}} \leq \kappa_1 \int_0^h g(t)\mathrm{d}t,$$

where h plays the role of N^{-1} for us. Thus, in the best case, there exists $0<\gamma<1$ and $\kappa_2>0$ such that

$$\int_0^{N^{-1}} g(t)\mathrm{d}t \leq \kappa_2 N^{-\gamma}.$$

So, we just have to take the regularity of J such that $m > \frac{\gamma}{1-\gamma}$ to get a better order of convergence, since $\frac{m}{m+1} > \gamma$. But we obtain the same convergence order as in Debbar et al. (2016).

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