Journal of Applied Mathematics and Computing
https://doi.org/10.1007/s12190-024-02022-6

ORIGINAL RESEARCH

A novel conformable fractional approach to the Brusselator system with numerical simulation

Mohamed Lamine Merikhi[1] · **Hamza Guebbai**[1] · **Noureddine Benrabia**[1,2] · **Mohamed Moumen Bekkouche**[1,3]

Received: 7 November 2023 / Revised: 7 November 2023 / Accepted: 9 February 2024

Abstract
In this study, we delve into a comprehensive analysis of the Brusselator system, combining both analytical and numerical approaches. In summary, our initial approach involves revisiting the classic Brusselator system using a conformable fractional derivative-based approach. Starting from this innovative reformulation, we obtain a nonlinear Volterra-type equation. This transformation allows us to simultaneously demonstrate the existence and uniqueness of the solution, while providing us with the necessary tools to develop an efficient numerical approximation method to solve the problem. Subsequently, we present a numerical simulation based on the Nyström method.

Keywords Brusselator system · Conformable fractional derivatives · Volterra-type nonlinear equation · Nyström method

Mathematics Subject Classification 86A17 · 86A15 · 45D05 · 65D30

✉ Mohamed Lamine Merikhi
laminemerrikhi@gmail.com ; merikhi.mohamedlamine@univ-guelma.dz

Hamza Guebbai
guebaihamza@yahoo.fr ; hamza.guebbai@univ-guelma.dz

Noureddine Benrabia
n.benrabia@univ-soukahras.dz ; noureddinebenrabia@yahoo.fr

Mohamed Moumen Bekkouche
moumen-med@univ-eloued.dz ; moumen39m@gmail.com

1 Laboratoire de Mathématiques Appliquées et de Modélisation, Faculté de Mathématiques et de l'Informatique et des Sciences de la Matière, Université 8 Mai 1945 Guelma, B.P. 401, 24000 Guelma, Algeria

2 Faculté des Sciences et de la Technologie, Université Mohamed-Chérif Messaadia Souk Ahras, 41000 Souk Ahras, Algeria

3 Faculté des Sciences, Université El Oued, 39000 El Oued, Algeria

Published online: 12 March 2024

1 Introduction

Nature, an inexhaustible source of inspiration for scientists, often reveals its deepest complexity in the biological domain. Biological phenomena offer a fascinating display of dynamic and sometimes unpredictable behaviors. Among the most remarkable examples of this complexity is the Brusselator model, which originated from the Belousov–Zhabotinsky (BZ) system [1–3].

The Brusselator model [4], an elegant and powerful mathematical tool, plays a central role in understanding oscillatory phenomena and dynamic behaviors observed in various scientific fields. Initially developed to model chemical variations in the Belousov–Zhabotinsky (BZ) system, it quickly expanded its range of applications. Its versatility and utility make it an instrument of choice for exploring a variety of complex phenomena, finding significant applications in diverse fields such as biology, including plant growth, cloud formation, and disease propagation [5–7].

It is characterized by this type of chemical reactions:

$$\begin{cases} A & \overset{V_1}{\longmapsto} & X, \\ B+X & \overset{V_2}{\longmapsto} & Y+D, \\ 2X+Y & \overset{V_3}{\longmapsto} & 3X, \\ X & \overset{V_4}{\longmapsto} & E. \end{cases}$$

The Brusselator system involves two reactants, denoted as A and B, which interact with each other to produce the chemical products X and Y. The parameters V_1 and V_2 represent the rate constants associated with reactions involving reactants A and B, respectively. These rate constants determine the speed at which the reactants are converted into products. The parameters V_3 and V_4 are also rate constants, but they are associated with reactions involving the chemical products X and Y, respectively. Thanks to its differential equations,

$$\begin{cases} \frac{d}{dt}[X] = V_1[A] - V_2[B][X] + V_3[X]^2[Y] - V_4[X], \\ \frac{d}{dt}[Y] = V_2[B][X] - V_3[X]^2[Y], \\ [X](0) = [Y](0) = 0, \end{cases}$$

the model allows for representing chemical concentration fluctuations over time. Its utility extends far beyond chemistry, providing a mathematical framework to model and comprehend complex biological phenomena. As a result, researchers can analyze the underlying mechanisms of oscillatory behaviors, predict their evolution, and potentially intervene to gain a better understanding of and influence biological processes.

Due to the significance of the Brusselator model, numerous mathematicians are studying it comprehensively from both analytical and numerical perspectives.

Among the researchers who have developed advanced methods, some notable contributions are found in articles that have paved the way for precise numerical simulations of the Brusselator's behavior under various conditions, such as "Numerical Simulation of the Brusselator Model" by Wang et al [8] and "Efficient Numerical Methods for the Brusselator Model" by Zhang et al [9]. These advances have enabled the accurate prediction of the system's responses to different configurations and parameters.

Analytically, researchers have examined the stability of the Brusselator, thus laying the foundations for our understanding of the system's oscillatory behaviors. The article "Stability Analysis of the Brusselator Model" by Thomas and Glansdorff [10] is a notable example of these contributions. Additionally, studies on the system's dynamics, such as "Dynamical Behavior of the Brusselator Model" by Cross and Rasmussen, have allowed for a detailed exploration of the Brusselator's oscillation regimes [11–18].

While our research methodology is centered on a comprehensive study of this model, both from an analytical and numerical standpoint, using the Caputo-Fabrizio conformable fractional derivative [19] as an analytical tool, we would like to emphasize that, for our current work, it can be defined as follows:

for $\alpha \in [0, 1]$ et $t \in [0, T]$, $\forall Z \in C^1 [0, T]$,

$$D_{CF}^{\alpha} Z(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t Z'(s) \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right) ds. \tag{1}$$

Classical fractional derivatives, such as Riemann–Liouville and Caputo derivatives, are mathematical operators of great importance. They extend the traditional concept of differentiation by using fractional exponents [20]. Their utility is particularly remarkable in modeling complex phenomena such as diffusion, growth, and other nonlinear processes. One of their significant advantages lies in their ability to account for memory effects, meaning they take into consideration the historical data, which is essential for describing dynamic systems.

However, it is important to note that these fractional derivatives are not without drawbacks. For example, the kernel exhibits singularities, which can complicate their use in certain cases. Moreover, one peculiarity of the Riemann–Liouville derivative is that the derivative of a constant is not zero, which can pose issues for physical interpretation [21].

Indeed, there are certainly other fractional derivatives that make sense and are conformable. These fractional derivatives have been developed to address the limitations of Riemann–Liouville and Caputo fractional derivatives. They are generally easier to use and interpret than classical fractional derivatives

In other words, they achieve:

$$\lim_{\alpha \to 0} D^{\alpha} f(t) = f(t)$$
$$\lim_{\alpha \to 1} D^{\alpha} f(t) = f'(t)$$

Several authors have contributed to the advancement of conformable fractional derivatives, including Khalil [22], Caputo and Fabrizio [19], and Guebai and Ghiat [23].

In 2015, Michele Caputo and Mauro Fabrizio introduced their conformable derivative (1) in such a way that $\alpha = 0$ has limited relevance, and $\alpha = 1$ corresponds to the standard derivative and holds significant physical importance as it describes velocity.

We have also chosen this derivative because it preserves the asymptotic approximation of the derivative. This is particularly relevant when dealing with non-locality and real-time evolution, as it introduces a memory effect without compromising the fundamental principles associated with the physical necessity of the classical derivative. This decision is also based on the presence of the integral term in the lemma, ensuring a complete history of the problem and creating a memory effect. Furthermore, the choice of this derivative maintains a nonsingular kernel, thus ensuring system stability.

In summary, our approach begins by taking the classic Brusselator system and reformulating it using a conformable fractional derivative. This new formulation allows us to convert it into a nonlinear Volterra-type equation, which enables us to prove the existence and uniqueness of the solution while providing us with the necessary tools to develop a numerical approximation method to solve the problem. We further support this approach with a numerical simulation based on the Nyström method.

2 Fractional model

Initially, by simplifying the notation of our system of equations, we are going to equate the components X, Y, A, B, to their respective concentrations $[X]$, $[Y]$, $[A]$, $[B]$, yielding: For $T \gg 1$ and $t \in [0, T]$,

$$X'(t) = V_1 A - V_2 B X(t) + V_3 X^2(t) Y(t) - V_4 X(t), \tag{2}$$

$$Y'(t) = V_2 B X(t) - V_3 X^2(t) Y(t), \tag{3}$$

$$X(0) = Y(0) = 0. \tag{4}$$

Our initial approach aims to reduce the number of unknowns. In this regard, we observe that the general solution of the equation Y' in (3) is given for all values of $t \in [0, T]$ as

$$Y(t) = \left(V_2 B \int_0^t X(s) \exp\left(-V_3 \int_s^0 X^2(\theta) d\theta \right) ds + c \right) \exp\left(-V_3 \int_0^t X^2(\theta) d\theta \right).$$

However, taking into account the initial condition $Y(0) = 0$, we obtain:

$$\forall t \in [0, T], \quad Y(t) = V_2 B \int_0^t X(s) \exp\left(-V_3 \int_s^t X^2(\theta) d\theta \right) ds. \tag{5}$$

By substituting (5) into (2), we obtain for all values of $t \in [0, T]$:

$$\begin{aligned} X'(t) = {} & V_1 A - (V_4 + V_2 B)\, X(t) \\ & + V_3 V_2 B X^2(t) \int_0^t X(s) \exp\left(-V_3 \int_s^t X^2(\theta) d\theta \right) ds\,. \end{aligned}$$

The new perspective in our work is to introduce $D_{CF}^{\alpha}X(t)$ in place of $X'(t)$, which results in the following fractional Brusselator model:

$$D_{CF}^{\alpha}X(t) = V_1 A - (V_4 + V_2 B) X(t) + V_3 V_2 B X^2(t) \int_0^t X(s) \exp\left(-V_3 \int_s^t X^2(\theta) d\theta\right) ds$$

Using the fractional integration formula developed in [24], which states that:

$$\forall t \in [0, T], \quad \forall Z \in C^1[0, T], \quad D_{CF}^{\alpha}Z(t) = F(t, Z(t)),$$

is equivalent to:

$$\forall t \in [0, T], \quad Z(t) = \left(\frac{\alpha}{M(\alpha)}\right) \int_0^t F(s; Z(s)) ds + \frac{1-\alpha}{M(\alpha)} F(t, Z(t)),$$

we obtain, $\forall t \in [0, T]$,

$$\begin{aligned}
&\left(1 + \left(\frac{1-\alpha}{M(\alpha)}\right)(V_4 + V_2 B)\right) X(t) \\
&= \left(\frac{1-\alpha}{M(\alpha)}\right)(V_3 V_2 B) X^2(t) \int_0^t X(s) \exp\left(-V_3 \int_s^t X^2(\theta) d\theta\right) ds \\
&\quad + \left(\frac{\alpha}{M(\alpha)}\right) V_1 A t \\
&\quad - \left(\frac{\alpha}{M(\alpha)}\right)(V_4 + V_2 B) \int_0^t X(s) ds \\
&\quad + \left(\frac{\alpha}{M(\alpha)}\right)(V_3 V_2 B) \int_0^t X^2(s) \left(\int_0^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta\right) ds.
\end{aligned} \tag{6}$$

As α approaches 1, and using the velocity parameters V_2 and V_3 near 0, the term below becomes negligible, i.e.

$$\left(\frac{1-\alpha}{M(\alpha)}\right)(V_3 V_2 B) X^2(t) \int_0^t X(s) \exp\left(-V_3 \int_s^t X^2(\theta) d\theta\right) ds \approx 0,$$

This provides the final form of our problem as follows:

Find $X \in C^0[0, T]$, the Banach space of continuous functions on $[0, T]$ in $\mathbb{R}$, equipped with its usual norm $\forall Z \in C^0[0, T], \quad | Z | = \max_{t \in [0,T]} | Z(t) |$,

such that

$$\forall t \in [0, T], \quad X(t) = \int_0^t N[X](s) ds + G(t),$$

where, $N : C^0[0,T] \longrightarrow C^0[0,T]$ and $\forall X \in C^0[0,T]$, $\forall s \in [0,T]$,

$$X \longmapsto N[X](s) = \mu_\alpha X(s) + \eta_\alpha X^2(s) \int_0^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta,$$

and

$$G(t) = \frac{\alpha V_1 A}{M(\alpha) + (1-\alpha)(V_4 + V_2 B)} t,$$

with,

$$\mu_\alpha = -\frac{\alpha (V_4 + V_2 B)}{M(\alpha) + (1-\alpha)(V_4 + V_2 B)},$$
$$\eta_\alpha = \frac{\alpha (V_2 V_3 B)}{M(\alpha) + (1-\alpha)(V_4 + V_2 B)}.$$

Emphasizing the obvious fact that $M(\alpha) + (1-\alpha)(V_4 + V_2 B) \neq 0$.

3 Analytical study

For $[\beta, \gamma] \subset [0,T]$ and $a, b \in \mathbb{R}$, we introduce the set

$$B_{a,b}[\beta,\gamma] = \left\{X \in C^0[\beta,\gamma] : \forall s \in [\beta,\gamma],\ \ a \leq X(s) \leq b\right\}.$$

In the following, we consider that X is extended by 0 over $[0,T] \backslash [\beta,\gamma]$. This gives:

$$N[X](s) = \begin{cases} \mu_\alpha X(s) + \eta_\alpha X^2(s) \int_\beta^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta, & s \in [\beta,\gamma], \\ 0, & s \in [0,T] \setminus [\beta,\gamma]. \end{cases} \tag{7}$$

Theorem 1 $a, b \in \mathbb{R},\ a < b,\ \forall X, Y \in B_{a,b}[\beta,\gamma],\ \ \exists L_{a,b} > 0$

$$\| N[X] - N[Y] \|_{C^0[\beta,\gamma]} \leq L_{a,b} \| X - Y \|_{C^0[\beta,\gamma]}$$

Proof Since $| N[X](s) - N[Y](s) | = 0$ for $s \in [0,\beta] \cup [\gamma, T]$, we estimate only on $[\beta,\gamma]$:

$$\begin{aligned} \forall s \in [\beta,\gamma],\ \ | N[X](s) - N[Y](s) | = | \mu_\alpha (X(s) - Y(s)) & \\ + \eta_\alpha \left(X^2(s) \int_\beta^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta \right. & \\ \left. - Y^2(s) \int_\beta^s Y(\theta) \exp\left(-V_3 \int_\theta^s Y^2(\tau) d\tau\right) d\theta \right) | & \end{aligned}$$

This gives, $\forall s \in [\beta, \gamma]$,

$$
\begin{aligned}
&\mid N[X](s) - N[Y](s) \mid \leq \mid \mu_\alpha \mid\mid X(s) - Y(s) \mid \\
&+ \overbrace{\mid \eta_\alpha \mid\mid X^2(s) - Y^2(s) \mid\mid \int_\beta^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta \mid}^{I_1} \\
&+ \overbrace{\mid \eta_\alpha \mid\mid Y(s) \mid^2 \mid \int_\beta^s X(\theta) \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta - \int_\beta^s Y(\theta) \exp\left(-V_3 \int_\theta^s Y^2(\tau) d\tau\right) d\theta \mid}^{I_2}.
\end{aligned} \tag{8}
$$

It is clear that for $\forall X \in B_{a,b}[\beta, \gamma]$, $\| X \|_{C^0[\beta,\gamma]} \leq \max(\mid a \mid, \mid b \mid)$, which implies,

$$
\begin{aligned}
I_1 &\leq 2 \mid \eta_\alpha \mid (max(\mid a \mid, \mid b \mid))^2 s \mid X(s) - Y(s) \mid, \\
&\leq 2 \mid \eta_\alpha \mid (max(\mid a \mid, \mid b \mid))^2 T \| X - Y \|_{C^0[\beta,\gamma]}.
\end{aligned}
$$

And,

$$
\begin{aligned}
I_2 \leq &\mid \eta_\alpha \mid \left(max\left(a^2, b^2\right)\right) \int_\beta^s \mid X(\theta) - Y(\theta) \mid \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) d\theta \\
&+ \mid \eta_\alpha \mid \left(max(\mid a \mid, \mid b \mid)^2\right) \int_\beta^s \mid Y(\theta) \mid \times \\
&\underbrace{\mid \exp\left(-V_3 \int_\theta^s X^2(\tau) d\tau\right) - \exp\left(-V_3 \int_\theta^s Y^2(\tau) d\tau\right) \mid}_{I_3} d\theta.
\end{aligned}
$$

But we know that for $\forall t_1, t_2 \in \mathbb{R}_+$; $\mid e^{-t_1} - e^{-t_2} \mid \leq \mid t_1 - t_2 \mid$. Therefore,

$$
\begin{aligned}
I_3 &\leq V_3 \mid \int_\theta^s \left(X^2(\tau) - Y^2(\tau)\right) d\tau \mid, \\
&\leq 2V_3 \max(\mid a \mid, \mid b \mid) T \| X - Y \|_{C^0[\beta,\gamma]}.
\end{aligned}
$$

This gives,

$$
\begin{aligned}
I_2 \leq &\mid \eta_\alpha \mid \left(max\left(a^2, b^2\right)\right) T \| X - Y \|_{C^0[\beta,\gamma]} \\
&+ 2V_3 \mid \eta_\alpha \mid \left(max\left(a^4, b^4\right)\right) T^2 \| X - Y \|_{C^0[\beta,\gamma]}
\end{aligned}
$$

Substituting I_1 and I_2 into (10), we find,

$$\begin{aligned}| N[X](s) - N[Y](s) | \leq & \mid \mu_\alpha \mid || X - Y ||_{C^0[\beta,\gamma]} \\ & + 2 \mid \eta_\alpha \mid \left(max \left(a^2, b^2 \right) \right) T \; || X - Y ||_{C^0[\beta,\gamma]} \\ & + 2V_3 \mid \eta_\alpha \mid \left(max \left(a^4, b^4 \right) \right) T^2 \; || X - Y ||_{C^0[\beta,\gamma]} \, .\end{aligned}$$

We conclude that, $\forall X, Y \in B_{a,b} \left[\beta, \gamma \right]$,

$$|| N[X] - N[Y] ||_{C^0[\beta,\gamma]} \leq L_{a,b} \; || X - Y ||_{C^0[\beta,\gamma]},$$

with,

$$L_{a,b} = \left(\mid \mu_\alpha \mid + 2T \mid \eta_\alpha \mid \left(\max \left(a^2, b^2 \right) \right) + 2V_3 T^2 \mid \eta_\alpha \mid \left(\max \left(a^4, b^4 \right) \right) \right).$$

□

Theorem 2 *: The equation*

$$X(t) = \int_0^t N[X](s) \, ds + G(t) \, .$$

has a unique continuous solution X in $C^0[0, T]$.

Proof Let $T_1 \in]0, T]$, sufficiently close to zero. For $t \in [0, T_1]$, we define the Picard sequence $\{X_n\}_{n \geq 0}$ [25] as follows:

$$\begin{cases} X_0(t) = G(t), \\ X_{n+1}(t) = \int_0^t N[X_n](s) \, ds + G(t), \quad n \geq 0. \end{cases} \tag{9}$$

Now, if $Z \in C^0[0, T_1]$, such that

$$\|Z - G\|_{C^0[0,T_1]} \leq \frac{1}{1-\delta} \|M[G]\|_{C^0[0,T_1]},$$

where $\delta \in]0, 1[$ and $M[G](t) = \int_0^t N[G](s) \, ds$.

This results in $\forall s \in [0, T_1]$,

$$-\frac{1}{1-\delta} \; || M[G] ||_{C^0[0,T_1]} \leq Z(s) - G(s) \leq \frac{1}{1-\delta} \; || M[G] ||_{C^0[0,T_1]},$$

$$a = -\frac{1}{1-\delta} \; || M[G] ||_{C^0[0,T_1]} + \min_{0 \leq s \leq T_1} G(s) \leq Z(s) \leq$$

$$\frac{1}{1-\delta} \; || M[G] ||_{C^0[0,T_1]} + \max_{0 \leq s \leq T_1} G(s) = b.$$

This leads to,
$\exists L_{a,b} > 0, \forall X, Y \in B_{a,b}[0, T_1],$

$$|| N[X] - N[Y] ||_{C^0[0,T_1]} \leq L_{a,b} || X - Y ||_{C^0[0,T_1]} .$$

We choose T_1 sufficiently close to 0 to obtain

$$T_1 L_{a,b} \leq \delta.$$

We introduce the mathematical statement for $n \geq 1$

$$(P_n) \begin{cases} || X_n - G ||_{C^0[0,T_1]} \leq \dfrac{1}{1-\delta} \|M[G]\|_{C^0[0,T_1]}, \\ || X_{n+1} - X_n ||_{C^0[0,T_1]} \leq \delta^n || X_1 - X_0 ||_{C^0[0,T_1]} . \end{cases} \tag{10}$$

We proceed by induction, we have, $\forall t \in [0, T_1],$

$$X_1(t) = \int_0^t N[G](s)ds + G(t),$$
$$| X_1(t) - G(t) | \leq \max_{0 \leq t \leq T_1} | \int_0^t N[G](s)ds |,$$
$$|| X_1 - G ||_{C^0[0,T_1]} \leq \frac{1}{1-\delta} \|M[G]\|_{C^0[0,T_1]},$$

This allows us to perform, $\forall t \in]0, T_1]$

$$\begin{aligned} | X_2(t) - X_1(t) | &= | \int_0^t (N[X_1](s) - N[X_0](s)) ds |, \\ &\leq T_1 L_{a,b} || X_1 - X_0 ||_{C^0[0,T_1]}, \\ &\leq \delta || X_1 - X_0 ||_{C^0[0,T_1]} . \end{aligned}$$

Now, suppose that (P_n) is true and let's demonstrate (P_{n+1})

$$\begin{aligned} || X_{n+1} - G ||_{C^0[0,T_1]} &\leq || X_{n+1} - X_n ||_{C^0[0,T_1]} + || X_n - X_{n-1} ||_{C^0[0,T_1]} \\ &+ ...+ || X_1 - X_0 ||_{C^0[0,T_1]}, \\ &\leq \delta^n || X_1 - X_0 ||_{C^0[0,T_1]} + \delta^{n-1} || X_1 - X_0 ||_{C^0[0,T_1]} \\ &+ ... + \delta^0 || X_1 - X_0 ||_{C^0[0,T_1]}, \\ &\leq \frac{1-\delta^{n+1}}{1-\delta} || X_1 - X_0 ||_{C^0[0,T_1]}, \\ &\leq \frac{1}{1-\delta} || M[G] ||_{C^0[0,T_1]} . \end{aligned}$$

This allows us to perform for all $\forall t \in [0, T_1]$

$$\begin{aligned} \mid X_{n+2}(t) - X_{n+1}(t) \mid &= \mid \int_0^t \left(N\left[X_{n+1}\right](s) - N\left[X_n\right](s) \right) ds \mid, \\ &\leq T_1 L_{a,b} \mid\mid X_{n+1} - X_n \mid\mid_{C^0[0,T_1]}, \\ &\leq \delta^{n+1} \mid\mid X_1 - X_0 \mid\mid_{C^0[0,T_1]} . \end{aligned}$$

Now, we demonstrate the existence of the solution by defining the sequence:

$$\forall n \geq 1, \quad X_n = \sum_{k=1}^{n} (X_k - X_{k-1}) + G. \tag{11}$$

But, $\sum_{k=1}^{n} (X_k - X_{k-1})$ is normaly convergent, i.e.

$$\begin{aligned} \mid\mid \sum_{k=1}^{n} (X_k - X_{k-1}) \mid\mid_{C^0[0,T_1]} &\leq \sum_{k=1}^{n} \mid\mid X_k - X_{k-1} \mid\mid_{C^0[0,T_1]}, \\ &\leq \left(\sum_{k=0}^{n-1} \delta^k \right) \mid\mid X_1 - X_0 \mid\mid_{C^0[0,T_1]}, \\ &\leq \frac{1}{1-\delta} \mid\mid X_1 - X_0 \mid\mid_{C^0[0,T_1]} . \end{aligned}$$

then, $\exists X \in C^0\,[0, T_1]$, $\lim\limits_{n \to +\infty} X_n = X$

$$\begin{aligned} \lim_{n \to +\infty} X_{n+1}(t) &= \lim_{n \to +\infty} \int_0^t N\left[X_n\right](s) ds + G(t), \\ \lim_{n \to +\infty} X_{n+1}(t) &= \int_0^t \lim_{n \to +\infty} N\left[X_n\right](s) ds + G(t), \\ \lim_{n \to +\infty} X_{n+1}(t) &= \int_0^t N\left[\lim_{n \to +\infty} X_n\right](s) ds + G(t). \end{aligned}$$

Therefore,

$$\forall t \in [0, T_1], \quad X(t) = \int_0^t N\left[X\right](s) ds + G(t).$$

We assume that over $[0, T_1]$, our equation has two solutions X and $\widetilde{X}$, with,

$$\forall t \in [0, T_1], \widetilde{X}(t) = \int_0^t N\left[\widetilde{X}\right](s) ds + G(t). \tag{12}$$

Similarly, we conclude that

$$|| X - \widetilde{X} ||_{C^0[0,T_1]} \leq \delta || X - \widetilde{X} ||_{C^0[0,T_1]} .$$

Therefore, $|| X - \widetilde{X} ||_{C^0[0,T_1]} = 0$. This proves the uniqueness of the solution X.

For $t > T_1$, we choose T_2 closer to T_1,in same way and, we rewrite our equation in the following form,

$$\forall t \in [T_1, T_2], \quad X(t) = \int_{T_1}^{t} N[X](s)ds + G_1(t), \tag{13}$$

where,

$$G_1(t) = \int_{T_1}^{T_2} N[X_1](s)ds + G(t),$$

and X_1 is the unique solution obtained previously on $[T_0, T_1]$. Using Theorem 1, which proves the local Lipschitz property of N and following the same steps as before, we show that (18) has a unique solution $X_2 \in C^0[T_1, T_2]$.

We notice that, $X_1(T_1) = X_2(T_2)$, which proves that

$$X_{12} = \begin{cases} X_1(t) & [0, T_1], \\ X_2(t) & [T_1, T_2], \end{cases}$$

belongs to $C^0[T_0, T_2]$ and it is the unique solution of our equation on $[T_0, T_2]$. We repeat this same process for the intervals $\left[T_i, T_{i+1}\right]$, $2 \leq i \leq N-1$, in order to construct the unique solution of our equation in $C^0[0, T]$. □

4 Numerical simulation

In this section, we propose a numerical simulation based on the Nyström method [25–28] with the trapezoidal rule [29, 30]. For that, we introduce a subdivision of $[0, T]$ as follows: $\forall n \geq 2, \quad h = \dfrac{T}{n+1}, \quad t_j = (j-1)h, \quad 1 \leq j \leq n.$

The weights of the numerical integration method are given for $1 \leq j \leq n$, by:

$$\begin{cases} W_1 = W_n = \dfrac{1}{2}, \\ W_j = 1, \quad 2 \leq j \leq n-1. \end{cases}$$

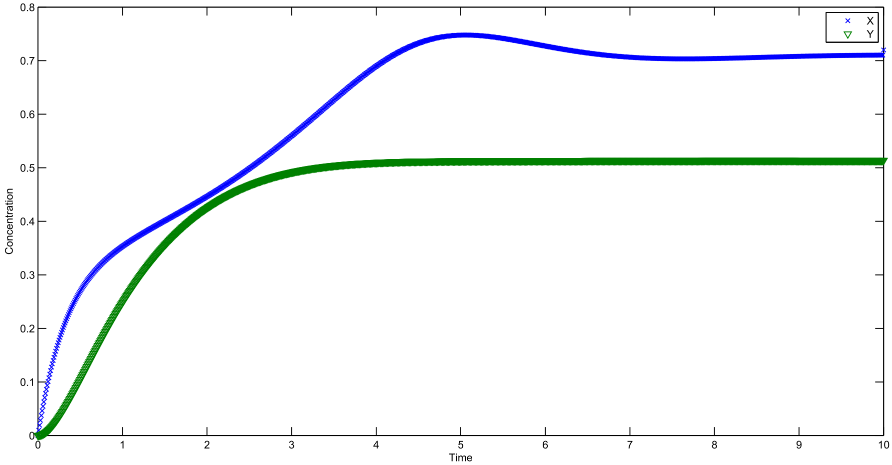

Fig. 1 X dominates Y

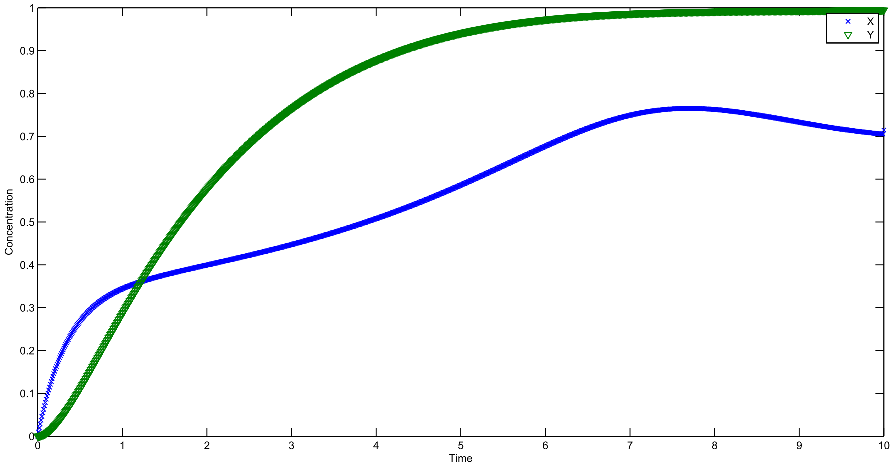

Fig. 2 Y dominates X

Once this method is applied to our equation, we obtain, for $n \geq 2$, the following nonlinear system:

$$\begin{aligned} X_1 &= 0, \\ X_i &= h \sum_{j=1}^{i} W_j \left(\mu_\alpha X_j + \eta_\alpha h X_j^2 \sum_{p=1}^{j} W_p X_p \exp \left(-V_3 h \sum_{q=p}^{j} W_q X_q^2 \right) \right) \\ &\quad + G(t_i), \quad 2 \leq i \leq n. \end{aligned}$$

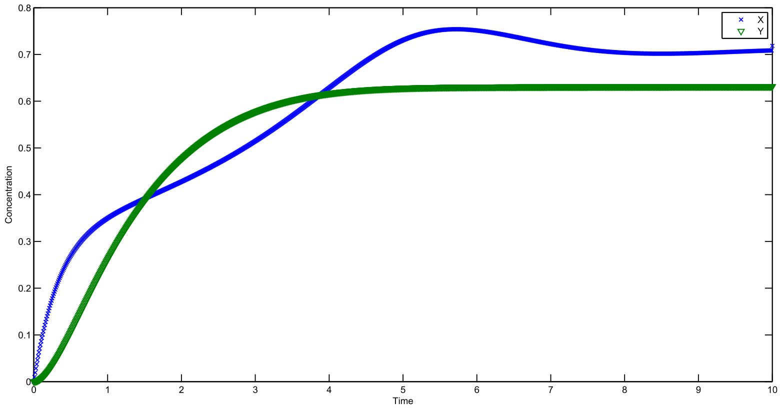

Fig. 3 No one dominates

Which we rewrite as

$$\begin{aligned} X_1 &= 0, \\ X_i &= \phi_i(X_i) + \psi_i, \quad 2 \le i \le n, \end{aligned}$$

where,

$$\begin{aligned} \phi_i(X_i) &= hW_i\left(\mu_\alpha X_i + \eta_\alpha h W_i X_i^3 \exp\left(-V_3 h W_i X_i^2\right)\right), \\ \psi_i &= h\sum_{j=1}^{i-1} W_j \left(\mu_\alpha X_j + \eta_\alpha h X_j^2 \sum_{p=1}^{j} W_p X_p \exp\left(-V_3 h \sum_{q=p}^{j} W_q X_q^2\right)\right) + G(t_i). \end{aligned}$$

For each $2 \le i \le n$, X_i is approached using a Banach sequence of the form

$$\begin{aligned} X_i^0 &= X_{i-1}, \\ X_i^{\nu+1} &= \phi_i(X_i^\nu) + \psi_i, \quad \nu \ge 0, \end{aligned}$$

with the following stop condition $\mid X_i^{\text{new}} - X_i^{\text{old}} \mid < 10^{-7}$.

The sequence $\{X_i\}_{i=1}^n$, once computed, will be used to approximate the function Y using

$$\begin{aligned} Y_1 &= 0, \\ Y_i &= V_2 B h \sum_{j=1}^{i} W_j X_j \exp\left(-V_3 h \sum_{p=j}^{i} W_p X_p\right), \quad 2 \le i \le n. \end{aligned}$$

If we take $A = 0.5$, $B = 0.7$, $V_1 = 2$, $V_2 = 2$, $V_3 = 2.7$, $V_4 = 1.4$, $\alpha = 0.99999$, $T = 10$ and $n = 1000$, we obtain a situation where X dominates Y, see Fig. 1.

If we take $A = 0.5$, $B = 0.7$, $V_1 = 2$, $V_2 = 2$, $V_3 = 1.4$, $V_4 = 1.4$, $\alpha = 0.99999$, $T = 10$ and $n = 1000$, we obtain a situation where Y dominates X, see Fig. 2.

If we take $A = 0.5$, $B = 0.7$, $V_1 = 2$, $V_2 = 2$, $V_3 = 2.2$, $V_4 = 1.4$, $\alpha = 0.99999$, $T = 10$ and $n = 1000$, we obtain a situation where no one dominates, see Fig. 2.

Data availability Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

References

1. Belousov, B.P.: A Reaction with a Periodic Variation of the Colour. Collection of Abstracts of Scientific Papers, pp. 145–147. Institute of Biological Physics, Academy of Sciences of the USSR, Moscow (1959)
2. Manohara, G., Kumbinarasaiah, S.: Fibonacci wavelet collocation method for the numerical approximation of fractional order Brusselator chemical model. J. Math. Chem. (2023). https://doi.org/10.1007/s10910-023-01521-4
3. Sarwar, S., Iqbal, S.: Stability analysis, dynamical behavior and analytical solutions of nonlinear fractional differential system arising in chemical reaction. Chin. J. Phys. **56**, 374–384 (2018). https://doi.org/10.1016/j.cjph.2017.11.009
4. Kundepudi, D., Prigogine, I.: Modern Thermodynamics: From Heat Engines to Dissipative Structures. Wiley, New York (1998)
5. Fang, Y., Wang, H.: A Brusselator-based model for the growth of a single cell under nutrient limitation. Sci. Rep. **9**(1), 1–10 (2019)
6. Tsuda, T., Mori, H.: A coupled Brusselator model for the formation of clouds and rain. J. Atmos. Sci. **51**(16), 2737–2749 (1994)
7. Liu, Y., Li, W.: A Brusselator model for the spatial spread of epidemics. J. Theor. Biol. **298**, 106–112 (2012)
8. Nonlinear stability analysis of the full Brusselator reaction–diffusion model. https://link.springer.com/article/10.1134/S0040579514060025
9. Manaa, S.A., Saeed, R.K., Easif, F.H.: Numerical stability of Brusselator system. Raf. J. Comput. Math. **8**, 2 (2011)
10. Goryunov, V.E.: The Andronov–Hopf bifurcation in a biophysical model of the Belousov reaction. Aut. Control Comput. Sci. **52**, 694–699 (2018). https://doi.org/10.3103/S0146411618070118
11. Sukhtayev, A., Zumbrun, K., Jung, S., et al.: Diffusive stability of spatially periodic solutions of the Brusselator model. Commun. Math. Phys. **358**, 1–43 (2018). https://doi.org/10.1007/s00220-017-3056-x
12. Prigogine, I., Lefever, R.: Symmetry breaking in irreversible processes. J. Chem. Phys. **54**(12), 4648–4654 (1971)
13. Mathematical Modeling of the Brusselator. https://www3.nd.edu/~powers/mcdowell.pdf
14. Kaplan, D., Nijhout, H.F.: Oscillatory behavior in a model of the Belousov–Zhabotinsky reaction. J. Chem. Phys. **61**(12), 4994–5008 (1976)
15. Gurevich, Y.M., Melnikov, A.V.: Oscillations in a model of the Belousov–Zhabotinsky reaction. Physica D **1**(1), 1–19 (1978)
16. Cross, M.C., Rasmussen, P.G.: Pattern formation in the Belousov–Zhabotinsky reaction. J. Chem. Phys. **69**(7), 3239–3250 (1978)
17. Grassberger, P., Mandelbrot, B.B.: The strange attractor of the Belousov–Zhabotinsky reaction. Phys. Lett. A **99**(1–2), 216–223 (1984)

18. Singh, P.: Applications of the Brusselator model to chemical, biological, and ecological systems. Prog. Theor. Phys. **70**(6), 1779–1792 (1983)
19. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Progr. Frat. Differ. Appl. **1**(2), 1–3 (2015). https://doi.org/10.12785/pfda/010201
20. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, New York (1993)
21. Atangana, A., Baleanu, D.: Fractional Variational Calculus with Applications in Mechanics. Springer (2017)
22. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
23. Guebbai, H., Ghiat, M.: New conformable fractional derivative definition for positive and increasing functions and its generalization. Adv. Dyn. Syst. Appl. **11**(2), 105–111 (2016)
24. Moumen Bekkouche, M., Guebbai, H., Kurulay, M., et al.: A new fractional integral associated with the Caputo–Fabrizio fractional derivative. Rend. Circ. Mat. Palermo II Ser. **70**, 1277–1288 (2021). https://doi.org/10.1007/s12215-020-00557-8
25. Linz, P.: Analytical and Numerical Methods for Volterra Equations. Society for Industrial Mathematics (1987)
26. Segni, S., Ghiat, M.: Hamza Guebbai new approximation method for volterra nonlinear integro-differential equation. https://doi.org/10.1142/S1793557119500165
27. Atkinson, K.E.: The Numerical Solution of Integral Equations of the Second Kind. Cambridge University Press, Cambridge (1997)
28. Atkinson, K., Han, W.: Theoretical Numerical Analysis: A Functional Analysis Approach. Springer, New York (2009)
29. Gautschi, W.: Numerical Analysis. Springer, New York (2012)
30. Tair, B., Ghait, M., Guebbai, H., Mohemd, A.Z.: Numerical solution of non-linear volterra integral equation of the first kind. Bol. Soc. Paran. Mat. **41**, 1–11 (2023)